

# DERIVED CATEGORIES AND LIE ALGEBRAS

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*Dedicated to Professor George Lusztig*

ABSTRACT. Let  $\mathcal{D}^b(A)$  be the derived category of a finite dimensional basic algebra  $A$  with finite global dimension. We construct the Lie algebra arising from the 2-periodic version  $\mathcal{K}_2(\mathcal{P}(A))$  of  $\mathcal{K}^b(\mathcal{P}(A))$  in term of constructible functions on varieties attached to  $\mathcal{K}_2(\mathcal{P}(A))$ .

## 1. INTRODUCTION

1.1. In the last thirty years of the twentieth century, there were two parallel fields in mathematics got extensively developed. One is the infinite dimensional Lie theory, in particular, the Kac-Moody Lie algebras. One is the representation theory of finite dimensional algebras, in particular, the representations of quivers. The close relation between the two subjects was discovered in a very early stage. Gabriel in [G] found that the quivers of finite representation type were given by the Dynkin graphs in Lie theory, and the dimension vectors provide the bijective correspondence between the isomorphism classes of indecomposable representations of the quiver and the positive root system of the semisimple Lie algebra. After the Gabriel theorem, a lot of progress on the connection between the representations of quivers or hereditary algebras and Lie algebras had been made, for example, by Bernstein-Gelfand-Ponomarev [BGP] and Dlab-Ringel [DR]. The final and most general result is the Kac theorem [K1] which extends to consider the quiver and the symmetric Kac-Moody algebra of arbitrary type. It states that the dimension vectors of indecomposable representations are exactly the positive roots, a unique indecomposable corresponds to each real root and infinitely many to each imaginary root; the multiplicity of the imaginary root, which is conjectured by Kac in [K2], is given by a geometric parameter in terms of representations of the quiver. A new progress on the Kac conjecture is by Crawley-Boevey and Van den Bergh in [CBV].

1.2. Ringel in [R2] discovered his Hall algebra structure by giving an answer to the following fundamental question: how to recover the underlying Lie algebra structure directly from the category of representations of the quiver.

Let  $Q$  be a quiver,  $A = \mathbb{F}_q Q$  the path algebra of  $Q$  over  $\mathbb{F}_q$ : the finite field with  $q$  elements. Set  $\mathcal{P} = \{\text{isoclasses of representations of } Q\}$ . For any  $\alpha \in \mathcal{P}$  choose  $V_\alpha$

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to be a representative in the class  $\alpha$ . Given three classes  $\lambda, \alpha, \beta \in \mathcal{P}$ , let  $g_{\alpha\beta}^\lambda$  be the order of the finite set  $\{W \triangleleft V_\lambda | W \cong V_\beta, V_\lambda/W \cong V_\alpha\}$ . By taking  $v = \sqrt{q}$  and the integral domain  $\mathbb{Q}(v)$ , the (twisted) Ringel-Hall algebra  $\mathcal{H}^*(A)$  can be defined to be a free  $\mathbb{Q}(v)$ -module with basis  $\{u_\lambda | \lambda \in \mathcal{P}\}$  and multiplication is given by

$$u_\alpha * u_\beta = v^{\langle \alpha, \beta \rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^\lambda u_\lambda \text{ for all } \alpha, \beta \in \mathcal{P}.$$

One may consider the subalgebra of  $\mathcal{H}^*(A)$  generated by  $u_i = u_{\alpha_i}$ , for  $i \in I (= Q_0)$  where  $\alpha_i \in \mathcal{P}$  is the isoclass of simple  $A$ -module at vertex  $i$ . The subalgebra is called the *composition algebra* and it is denoted by  $\mathcal{C}^*(A)$ . On the other hand, the index set  $I$  of simple  $A$ -modules together with the symmetric Euler form  $(I, (-, -))$  of  $A$  is a Cartan datum in the sense of Lusztig [L4]. For a Cartan datum  $(I, (-, -))$ , the quantized enveloping algebra  $\mathcal{U}_q$  defined by Drinfeld [Dr] and Jimbo [J] is associated with it. The positive part  $\mathcal{U}_q^+$  is generated by  $E_i, i \in I$  with subject to the quantum Serre relations.

There is a usual way to define the generic form  $\mathcal{C}^*(Q)$  of the composition algebra  $\mathcal{C}^*(A)$  by considering the representations of  $Q$  over infinitely many finite fields. Then  $\mathcal{C}^*(Q)$  is a  $\mathbb{Q}(v)$ -algebra where  $v$  becomes a transcendental element over  $\mathbb{Q}$ . Put  $u_i^{(*n)} = \frac{u_i^{*n}}{[n]_i!}$  for  $i \in I$  and  $n \in \mathbb{N}$  and let  $\mathcal{C}^*(Q)_\mathcal{Z}$  be the integral form of  $\mathcal{C}^*(Q)$ , which is generated by  $u_i^{(*n)}, i \in I, n \in \mathbb{N}$  over the integral domain  $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ . Also the quantum group  $\mathcal{U}_q^+$  has the integral form  $\mathcal{U}_\mathcal{Z}^+$ , which is generated by  $E_i^{(n)}, i \in I, n \in \mathbb{N}$  over  $\mathcal{Z}$ . Then by Ringel [R1] and Green [Gr], the canonical map  $\mathcal{C}^*(Q)_\mathcal{Z} \rightarrow \mathcal{U}_\mathcal{Z}^+$  by sending  $u_i^{(*n)}$  to  $E_i^{(n)}$  for  $i \in I$  and  $n \in \mathbb{N}$  leads to a  $\mathcal{Z}$ -algebra isomorphism, if the two algebras share a common Cartan datum.

Let  $\text{ind } \mathcal{P} = \{\text{isoclasses of indecomposable representations of } Q\}$ . Then  $g_{\alpha\beta}^\lambda$  can be regarded as a function on  $q$  for  $\alpha, \beta, \lambda \in \text{ind } \mathcal{P}$ . In fact, Ringel in [R2] proved that  $g_{\alpha\beta}^\lambda$  is an integral polynomial on  $q$  when  $Q$  is of finite type. One can take the integral value  $g_{\alpha\beta}^\lambda(1)$  by letting that  $q$  tends to 1. Ringel [R2], for  $Q$  of finite type, prove that the  $u_\alpha, \alpha \in \text{ind } \mathcal{P}$ , spanned a Lie subalgebra of  $\mathcal{C}^*(Q) |_{q=1}$  with Lie bracket

$$[u_\alpha, u_\beta] = \sum_{\lambda \in \text{ind } \mathcal{P}} (g_{\alpha\beta}^\lambda(1) - g_{\beta\alpha}^\lambda(1)) u_\lambda$$

for  $\alpha, \beta \in \text{ind } \mathcal{P}$ . This realized the positive part  $\mathfrak{n}^+$  of the semisimple Lie algebra  $\mathfrak{g}$ . Of course, Ringel's approach also works for  $Q$  of arbitrary type. In general, there exists the generic composition Lie subalgebra  $\mathcal{L}$  of  $\mathcal{C}^*(Q) |_{q=1}$  generated by  $u_i, i \in I$  and  $\text{ind } \mathcal{P}$  is no longer to index a basis of  $\mathcal{L}$ . Now  $\mathcal{L}$  is canonically isomorphic to the positive part  $\mathfrak{n}^+$  of the symmetric Kac-Moody Lie algebra  $\mathfrak{g}$ . For a realization of the whole  $\mathfrak{g}$ , not just its positive part, Peng and Xiao in [PX3] have constructed a Lie algebra from a triangulated category with the 2-periodic shift functor  $T$ , i.e,  $T^2 = 1$ . If, specially, consider the 2-periodic orbit category of the derived category of a finite dimensional hereditary algebra, the Lie algebra obtained in [PX3] gives rise to the global realization of symmetrizable Kac-Moody algebra of arbitrary type. In [PX3] they consider the triangulated categories over finite fields. Replacing counting the order of the filtration set, they calculate the order of the orbit space of a triangle. By a hard work, they obtained a Lie ring  $\mathfrak{g}_{(q-1)}$  over  $\mathbb{Z}/(q-1)$  for the prime powers  $q = |\mathbb{F}_q|$ . Then they performed their work over finite field extensions of

arbitrarily large order and construct a generic Lie algebra which is similar to the generic composition Lie subalgebra done by Ringel in [R3]. A transcendental Lie algebra was finally obtained.

1.3. Quickly after the work of Ringel [R1], people realized that a geometric setting of Ringel-Hall algebra is possible by using the convolution multiplication (see [Sch] and [L1]). Let  $Q$  be a quiver and  $\alpha = \sum_{i \in I} a_i i \in \mathbb{N}[I]$  a dimension vector. We fix a  $I$ -graded space  $\mathbb{C}^\alpha = (\mathbb{C}^{a_i})_{i \in I}$ . Then

$$\mathbb{E}_\alpha = \bigoplus_{h: s(h) \rightarrow t(h)} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{a_{s(h)}}, \mathbb{C}^{a_{t(h)}})$$

is an affine space. Set

$$G_\alpha = \prod_{i \in I} GL(a_i, \mathbb{C}).$$

For any  $(x_h) \in \mathbb{E}_\alpha$  and  $g = (g_i) \in G_\alpha$ , we define the action  $g \cdot (x_h) = (g_{t(h)} x_h g_{s(h)}^{-1})$ . For any  $Q$ -representation  $M$  with  $\dim M = \alpha$ , let  $\mathcal{O}_M \subset \mathbb{E}_\alpha$  be the  $G_\alpha$ -orbit of  $M$ .

For an algebraic variety  $X$  over  $\mathbb{C}$ , a subset  $A$  of  $X$  is said to be constructible if it is a finite union of locally closed subsets. A function  $f : X \rightarrow \mathbb{C}$  is constructible if it is a finite  $\mathbb{C}$ -linear combination of characteristic functions  $\mathbf{1}_{\mathcal{O}}$  for constructible subsets  $\mathcal{O}$ .

We define  $M_{G_\alpha}(Q)$  to be the space of constructible  $G_\alpha$ -invariant functions  $\mathbb{E}_\alpha \rightarrow \mathbb{C}$ , and let  $M_G(Q) = \bigoplus_{\alpha \in \mathbb{N}I} M_{G_\alpha}(Q)$ . Let  $\text{ind}\mathbb{E}_\alpha(Q)$  to be the constructible subset of  $\mathbb{E}_\alpha$  consisting of all points  $x$  which correspond to indecomposable  $Q$ -representations, and let  $\text{ind}M_{G_\alpha}(Q)$  to be the space of constructible  $G_\alpha$ -invariant functions over  $\text{ind}\mathbb{E}_\alpha$ . We may regard as  $\text{ind}M_{G_\alpha}(Q) = \{f \in M_{G_\alpha}(Q) | \text{supp} f \subseteq \text{ind}\mathbb{E}_\alpha\}$ , and  $\text{ind}M_G(Q) = \bigoplus_{\alpha \in R^+} \text{ind}M_{G_\alpha}(Q)$ , where, by Kac theorem,  $R^+$  is the positive root system of the Kac-Moody Lie algebra corresponding to  $Q$ . The space  $M_G(Q) = \bigoplus_{\alpha \in \mathbb{N}I} M_{G_\alpha}(Q)$  can be endowed with the associative algebra structure by the convolution multiplication:

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}(y) = \chi(\mathcal{F}_{\mathcal{O}_1 \mathcal{O}_2}^y)$$

for any  $G_\alpha$ -invariant constructible set  $\mathcal{O}_1$  and  $G_\beta$ -invariant constructible set  $\mathcal{O}_2$  with  $\alpha, \beta \in \mathbb{N}I$ , where  $\mathcal{F}_{\mathcal{O}_1 \mathcal{O}_2}^y = \{x \in \mathcal{O}_2 \mid M(x) \subseteq M(y) \text{ and } M(y)/M(x) \in \mathcal{O}_1\}$  and  $\chi(X)$  denotes the Euler characteristic of the topological space  $X$ . As in [Rie] and [DXX], it can be proved that the space  $\text{ind}M_G(Q)$  has a Lie algebra structure under the usual Lie bracket

$$[1_{\mathcal{O}_1}, 1_{\mathcal{O}_2}] = 1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} - 1_{\mathcal{O}_2} * 1_{\mathcal{O}_1}.$$

Applying this setup to the case  $Q$  being a tame quiver, Frenkel-Malkin-Vybornov [FMV] gave an explicit realization of the positive parts of affine Lie algebras.

1.4. The great progress is made by Lusztig, who apply the Hall algebras in a geometric setting to study the quantum groups (see [L1] and [L2]) and the enveloping algebras (see [L5]). The canonical bases of the quantum groups and the semicanonical bases of the enveloping algebras were originally constructed in terms of representations of quivers. However Lusztig [L2] has pointed out that a more suitable choice is the preprojective algebra, which is given by the double quiver of  $Q$  with the Gelfand-Ponomarev relations. Further progress in this direction is the study of Nakajima [N] on his quiver varieties, which leads to a geometric realization of the representation theory of Kac-Moody algebras.

Inspired by Ringel's work on Hall algebras and Lusztig's geometric approach to quantum groups, the aim of this paper is to give a global and geometric realization of the Lie algebras arising from the derived categories, which is a generalization of our earlier work [PX3].

1.5. If we consider the module category of  $A = \mathbb{C}Q/J$ , we have the algebraic variety  $\mathbb{E}_{\underline{d}}(Q, R)$  for  $A$ -modules with a fixed dimension vector  $\underline{d}$  and it is a  $G$ -variety where  $G = G_{\underline{d}}(Q)$  is a reductive group. According to the work of C.de Concini and E.Strickland in [CS] and M.Saorin and B.Huisgen-Zimmermann in [SHZ], this geometry can be generalized to over the chain complexes of  $A$ -modules. Section 2 is devoted to do this. Let  $K_0(\mathcal{D}^b(A))$  be the Grothendieck group of  $\mathcal{D}^b(A)$  and  $\underline{\dim}$  be the canonical map from the abelian group of dimension vector sequences to  $K_0(\mathcal{D}^b(A))$ . Given  $\mathbf{d} \in K_0(\mathcal{D}^b(A))$  and  $\underline{\mathbf{d}} \in \underline{\dim}^{-1}(\mathbf{d})$ , the set  $\mathcal{C}^b(A, \underline{\mathbf{d}})$  of all complexes of  $A$ -modules with the dimension vector sequences  $\underline{\mathbf{d}}$  and its subset  $\mathcal{P}^b(A, \underline{\mathbf{d}})$  of all projective complexes can be endowed with the affine variety structures. Let  $\mathcal{C}^b(A, \mathbf{d})$  (resp.  $\mathcal{P}^b(A, \mathbf{d})$ ) be the direct limit of  $\mathcal{C}^b(A, \underline{\mathbf{d}})$  (resp.  $\mathcal{P}^b(A, \underline{\mathbf{d}})$ ) for  $\underline{\mathbf{d}} \in \underline{\dim}^{-1}(\mathbf{d})$ . Here, we associate to  $\mathcal{K}^b(\mathcal{P}(A))$  its quotient space  $\mathcal{QP}^b(A, \mathbf{d})$  which is the direct limit of the quotient spaces  $\mathcal{QP}^b(A, \underline{\mathbf{d}})$  of  $\mathcal{P}^b(A, \underline{\mathbf{d}})$  under the action of some algebraic group  $G_{\underline{\mathbf{d}}}$ . Our main aims in Section 2 are to study the relation between  $\mathcal{Q}^b(A, \mathbf{d})$  and  $\mathcal{QP}^b(A, \mathbf{d})$ , the action of derived equivalence on  $\mathcal{Q}^b(A, \mathbf{d})$  and  $\mathcal{QP}^b(A, \mathbf{d})$  and characterize the orbit space  $\mathcal{QP}^b(A, \mathbf{d})$  of  $\mathcal{P}^b(A, \mathbf{d})$  under the action of the direct limit  $G_{\mathbf{d}}$  of algebraic groups  $G_{\underline{\mathbf{d}}}$ . Therefore the main point in Section 2 is that, we can regard the  $G_{\mathbf{d}}$ -invariant geometry in  $\mathcal{P}^b(A, \mathbf{d})$  as the moduli space in which the orbits index the isomorphism classes of objects in the derived category.

In Section 3, we consider the inverse limit of the  $\mathbb{C}$ -space of  $G_{\underline{\mathbf{e}}}$ -invariant constructible functions over  $\mathcal{P}^b(A, \underline{\mathbf{e}})$  for any  $\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})$ . Any element in the inverse limit can be viewed as a  $G_{\mathbf{d}}$ -invariant constructible function over  $\mathcal{P}^b(A, \mathbf{d})$ . We define the convolution between  $G_{\mathbf{d}}$ -invariant constructible functions. Our main result in Section 3 is that our convolution rule is well-defined. In order to prove it, we need to define the *naïve* Euler characteristic of the orbit spaces induced by triangles in the triangulated category as in [Jo1, Section 4.3]. The theorem of Rosenlicht [Ro] for the algebraic group action on varieties is crucial for us. Section 4 is just to transfer the results in Section 3 to the 2-periodic orbit categories of the derived categories.

Section 5 is devoted to verifying the Jacobi identity. In [PX3], by counting the Hall numbers  $F_{XY}^L$  for the triangles of the form  $X \rightarrow L \rightarrow Y \rightarrow X[1]$ , it has been proved that the Jacobi identity can be deduced from the octahedral axiom of the triangulated categories. However we need to prove that the correspondences among the various orbit spaces in the derived categories are actually given by the algebraic morphisms of algebraic varieties. We think this geometric method is more transparent to reflect the hidden symmetry in the derived category. Additionally, we get the two properties which is unknown in [PX3]. Firstly the proper assumption in [PX3] is not necessary, in fact, it is easy to give examples such that  $\underline{\dim} X = 0$  for some nonzero indecomposable  $X$  in  $\mathcal{D}^b(A)$ . Secondly we show that the Lie algebras arising from the 2-periodic orbit categories of the derived categories always possess the symmetric invariant form in the sense of Kac [K3], which is essentially non-degenerated. Section 6 is to apply the construction to the 2-periodic orbit categories of the derived categories of representations of quivers, particularly, tame quivers.

This gives rise to a global realization of the symmetric (generalized) Kac-Moody algebras of arbitrary type. In particular, an explicit realization of the affine Lie algebras.

1.6. Finally we should mention recent advances by Toën [T] and Joyce [Jo2]. Toën defined an associative algebra, called the derived Hall algebra associated to a dg category over a finite field. A direct proof for Toën's theorem is given in [XX]. Joyce considered a new Ringel-Hall type algebra consisting of functions over stacks associated to abelian categories. Their results can be viewed as improvements of the Ringel-Hall type algebra with respect to categorification and geometrization. However, it is unknown how to define an analogue of the derived Hall over the complex field (see [L6] [N]) or an analogue of the derived algebra for the 2-period version of a derived category (see [T] and [XX]). Hence, it is still an open question to define an associative multiplication which induces the Lie bracket in this paper and supplies the realization of the corresponding enveloping algebra.

## 2. TOPOLOGICAL SPACES ATTACHED TO DERIVED CATEGORIES

2.1. **Module varieties.** Given an associative algebra  $A$  over the complex field  $\mathbb{C}$ , in this paper, we always assume that  $A$  is both finite dimensional and finite global dimensional. By a result of P. Gabriel ([G]) the algebra  $A$  is given by a quiver  $Q$  with relations  $R$  (up to Morita equivalence). Let  $Q = (Q_0, Q_1, s, t)$  be a quiver, where  $Q_0$  and  $Q_1$  are the sets of vertices and arrows, respectively, and  $s, t : Q_1 \rightarrow Q_0$  are maps such that any arrow  $\alpha$  starts at  $s(\alpha)$  and terminates at  $t(\alpha)$ . For any dimension vector  $\underline{d} = (d_i)_{i \in Q_0}$ , we consider the affine space over  $\mathbb{C}$

$$\mathbb{E}_{\underline{d}}(Q) = \bigoplus_{\alpha \in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{s(\alpha)}}, \mathbb{C}^{d_{t(\alpha)}})$$

Any element  $x = (x_{\alpha})_{\alpha \in Q_1}$  in  $\mathbb{E}_{\underline{d}}(Q)$  defines a representation  $(\mathbb{C}^{\underline{d}}, x)$  where  $\mathbb{C}^{\underline{d}} = \bigoplus_{i \in Q_0} \mathbb{C}^{d_i}$ . A relation in  $Q$  is a linear combination  $\sum_{i=1}^r \lambda_i p_i$ , where  $\lambda_i \in \mathbb{C}$  and  $p_i$  are paths of length at least two with  $s(p_i) = s(p_j)$  and  $t(p_i) = t(p_j)$  for all  $1 \leq i, j \leq r$ . For any  $x = (x_{\alpha})_{\alpha \in Q_1} \in \mathbb{E}_{\underline{d}}$  and any path  $p = \alpha_1 \alpha_2 \cdots \alpha_m$  in  $Q$  we set  $x_p = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_m}$ . Then  $x$  satisfies a relation  $\sum_{i=1}^r \lambda_i p_i$  if  $\sum_{i=1}^r \lambda_i x_{p_i} = 0$ . If  $R$  is a set of relations in  $Q$ , then let  $\mathbb{E}_{\underline{d}}(Q, R)$  be the closed subvariety of  $\mathbb{E}_{\underline{d}}(Q)$  which consists of all elements satisfying all relations in  $R$ . Any element  $x = (x_{\alpha})_{\alpha \in Q_1}$  in  $\mathbb{E}_{\underline{d}}(Q, R)$  defines in a natural way a representation  $M(x)$  of  $A = \mathbb{C}Q/J$  with  $\dim M(x) = \underline{d}$ , where  $J$  is the admissible ideal generated by  $R$ . We consider the algebraic group

$$G_{\underline{d}}(Q) = \prod_{i \in Q_0} GL(d_i, \mathbb{C}),$$

which acts on  $\mathbb{E}_{\underline{d}}(Q)$  by  $(x_{\alpha})^g = (g_{t(\alpha)} x_{\alpha} g_{s(\alpha)}^{-1})$  for  $g \in G_{\underline{d}}$  and  $(x_{\alpha}) \in \mathbb{E}_{\underline{d}}$ . It naturally induces the action of  $G_{\underline{d}}(Q)$  on  $\mathbb{E}_{\underline{d}}(Q, R)$ . The induced orbit space is denoted by  $\mathbb{E}_{\underline{d}}(Q, R)/G_{\underline{d}}(Q)$ . There is a natural bijection between the set  $\mathcal{M}(A, \underline{d})$  of isomorphism classes of  $\mathbb{C}$ -representations of  $A$  with dimension vector  $\underline{d}$  and the set of orbits of  $G_{\underline{d}}(Q)$  in  $\mathbb{E}_{\underline{d}}(Q, R)$ . So we may identify  $\mathcal{M}(A, \underline{d})$  with  $\mathbb{E}_{\underline{d}}(Q, R)/G_{\underline{d}}(Q)$ .

**2.2. Categories of complexes.** First we consider the category of complexes  $\mathcal{C}(A)$ . Its objects are sequences  $M^\bullet = (M_n, \partial_n)$  of finite dimensional  $A$ -modules and their homomorphisms

$$(2.1) \quad \dots \xrightarrow{\partial_{n-1}} M_n \xrightarrow{\partial_n} M_{n+1} \xrightarrow{\partial_{n+1}} M_{n+2} \xrightarrow{\partial_{n+2}} \dots$$

such that  $\partial_{n+1}\partial_n = 0$  for all  $n$ . A morphism  $\phi^\bullet : M^\bullet \rightarrow M'^\bullet$  between two complexes is a sequence of homomorphisms  $\phi^\bullet = (\phi_n : M_n \rightarrow M'_n)_{n \in \mathbb{Z}}$  such that the following diagram is commutative.

$$(2.2) \quad \begin{array}{ccccccc} \dots & \xrightarrow{\partial_{n-1}} & M_n & \xrightarrow{\partial_n} & M_{n+1} & \xrightarrow{\partial_{n+1}} & M_{n+2} \xrightarrow{\partial_{n+2}} \dots \\ & & \phi_n \downarrow & & \phi_{n+1} \downarrow & & \phi_{n+2} \downarrow \\ \dots & \xrightarrow{\partial'_{n-1}} & M'_n & \xrightarrow{\partial'_n} & M'_{n+1} & \xrightarrow{\partial'_{n+1}} & M'_{n+2} \xrightarrow{\partial'_{n+2}} \dots \end{array}$$

One says that such a morphism is *homotopic to zero* if there are homomorphisms  $\sigma_n : M_n \rightarrow M'_{n-1}$  such that  $\phi_n = \sigma_{n+1}\partial_n + \partial'_{n-1}\sigma_n$  for all  $n \in \mathbb{Z}$ . The factor category  $\mathcal{K}(A)$  of  $\mathcal{C}(A)$  modulo the ideal of morphisms homotopic to zero is called the *homotopic category* of  $A$ -modules. For each  $n$  the *n-th homology* of a complex is defined as  $H_n(M^\bullet) = \text{Ker } \partial_n / \text{Im } \partial_{n-1}$ . Obviously, a morphism  $\phi^\bullet$  of complexes induces homomorphisms of homologies  $H_n(\phi^\bullet) : H_n(M^\bullet) \rightarrow H_n(M'^\bullet)$  and if  $\phi^\bullet$  is homotopic to zero, it induces zero homomorphisms of homologies. One call a morphism  $\phi^\bullet$  in  $\mathcal{C}(A)$  or in  $\mathcal{K}(A)$  *quasi-isomorphism* if the induced morphisms  $H_n(\phi^\bullet)$  are isomorphisms for all  $n$ . Now the derived category  $\mathcal{D}(A)$  is defined to be the category of fractions  $\mathcal{K}(A)[\mathcal{N}^{-1}]$ , where  $\mathcal{N}$  is the set of all quasi-isomorphisms, which is obtained from  $\mathcal{K}(A)$  by inverting all morphisms in  $\mathcal{N}$ . One calls a complex *right bounded* (*left bounded*, *bounded*, respectively) if there is  $n_0$  such that  $M_n = 0$  for  $n > n_0$  ( there is  $n_1$  such that  $M_n = 0$  for  $n < n_1$ , or there are both, respectively). The corresponding categories are denoted by  $\mathcal{C}^-(A)$ ,  $\mathcal{K}^-(A)$ ,  $\mathcal{D}^-(A)$  ( by  $\mathcal{C}^+(A)$ ,  $\mathcal{K}^+(A)$ ,  $\mathcal{D}^+(A)$ , or by  $\mathcal{C}^b(A)$ ,  $\mathcal{K}^b(A)$ ,  $\mathcal{D}^b(A)$ , respectively). In this paper we mainly deal with the bounded situation.

The category  $A\text{-mod}$  of finite dimensional  $A$ -modules can be naturally embedded into  $\mathcal{D}(A)$  (even in  $\mathcal{D}^b(A)$ ): a module  $M$  is identified with the complex  $M^\bullet$  such that  $M_0 = M$  and  $M_n = 0$  for  $n \neq 0$ .

A complex  $P^\bullet = (P_n, \partial_n)$  is called projective if all  $P_n$  are projective  $A$ -modules. Since the category  $A\text{-mod}$  has enough projective objects, one can replace, when considering right bounded homotopic and derived category, arbitrary complexes by projective ones. We denote by  $\mathcal{P}^-(A)$  and by  $\mathcal{P}^b(A)$  the full subcategories of  $\mathcal{C}^-(A)$  and  $\mathcal{C}^b(A)$  which consist of right bounded and bounded projective complexes, respectively. Actually, we have  $\mathcal{D}^-(A) \simeq \mathcal{K}^-(\mathcal{P}^-(A)) \simeq \mathcal{P}^-(A)/\mathcal{I}$ , where  $\mathcal{I}$  is the ideal of morphisms homotopic to zero (see[GM]). Moreover, every finite dimensional  $A$ -module  $M$  has a *projective cover*, i.e., an epimorphism  $p_M : P(M) \rightarrow M$  such that  $P(M)$  is projective and  $\text{Ker } p_M \subseteq \text{rad } P(M)$ , the radical of  $P(M)$ . Therefore, we can only consider *minimal or radical* projective complexes  $P^\bullet = (P_n, \partial_n)$  with the property:  $P_n$  is projective and  $\text{Im } \partial_n \subseteq \text{rad } P_{n+1}$  for all  $n$ . Let  $\text{rad } \mathcal{P}^-(A)$  be the full subcategory of  $\mathcal{P}^-(A)$  which consist of minimal projective complexes. Since every projective complex in  $\mathcal{P}^-(A)$  is quasi-isomorphic to a minimal projective complexes, we have  $\mathcal{D}^-(A) \simeq \text{rad } \mathcal{P}^-(A)/\mathcal{I}$ , where  $\mathcal{I}$  is the ideal of morphisms homotopic to zero. One immediately checks that a morphism  $\phi^\bullet$  between minimal

projective complexes induces an isomorphism in  $\mathcal{D}^-(A)$  if and only if  $\phi^\bullet$  itself is an isomorphism in  $\text{rad } \mathcal{P}^-(A)$ . If we further assume that the global dimension of  $A$  is finite, then we have  $\mathcal{D}^b(A) \simeq \text{rad } \mathcal{P}^b(A)/\mathcal{I}$ , since any bounded complex has a bounded projective resolution.

**2.3. Complex varieties.** The geometrization of  $A$ -modules with dimension vector  $\underline{d}$  can be carried over, in the same spirit, to the complexes. Let the algebra  $A = \mathbb{C}Q/J$  be of finite global dimension and the admissible ideal  $J$  is given by a set  $R$  of relations in  $Q$ . For a dimension vector  $\underline{d}$  we understand as  $\underline{d} : Q_0 \rightarrow \mathbb{N}$ . We set the  $Q_0$ -graded  $\mathbb{C}$ -space  $\mathbb{C}^{\underline{d}} = \bigoplus_{j \in Q_0} \mathbb{C}^{\underline{d}(j)}$ . For a sequence of dimension vectors  $\underline{d} = (\dots, \underline{d}_{-1}, \underline{d}_0, \underline{d}_1, \dots)$  with only finite many non-zero entries, we define  $\mathcal{C}^b(A, \underline{d})$  to be the closed subset of (see [SHZ])

$$\prod_{i \in \mathbb{Z}} \mathbb{E}_{\underline{d}_i}(Q, R) \times \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{C}}(V^{\underline{d}_i}, V^{\underline{d}_{i+1}})$$

which consists of elements  $(x_i, \partial_i)_i$ , where  $x_i \in \mathbb{E}_{\underline{d}_i}(Q, R)$  and  $M(x_i) = (\mathbb{C}^{\underline{d}_i}, x_i)$  is the corresponding  $A$ -module and  $\partial_i \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^{\underline{d}_i}, \mathbb{C}^{\underline{d}_{i+1}})$  is a  $A$ -module homomorphism from  $M(x_i)$  to  $M(x_{i+1})$  with the property  $\partial_{i+1}\partial_i = 0$ . In fact,  $(M(x_i), \partial_i)_i$ , or simply denoted by  $(x_i, \partial_i)_i$ , is a complex of  $A$ -modules and  $\underline{d}$  is called its dimension vector sequence.

The group  $G_{\underline{d}} := \prod_{i \in \mathbb{Z}} G_{\underline{d}_i}(Q)$  acts on  $\mathcal{C}^b(A, \underline{d})$  via the conjugation action

$$(g_i)_i(x_i, \partial_i)_i = ((x_i)^{g_i}, g_{i+1}\partial_i g_i^{-1})_i$$

where the action  $(x_i)^{g_i}$  was defined as in Section 2.1. Therefore the orbits under the action correspond bijectively to the isomorphism classes of complexes of  $A$ -modules.

We fix a set  $P_1, P_2, \dots, P_l$  to be a complete set of indecomposable projective  $A$ -modules (up to isomorphism). Let  $\mathcal{P}^b(A)$  be the full subcategory of  $\mathcal{C}^b(A)$  which consists of projective complexes  $P^\bullet = (P^i, \partial_i)$  such that each  $P^i$  has the decomposition  $P^i \cong \bigoplus_{j=1}^l e_j^i P_j$ . We denote by  $\underline{e}(P^i)$  the vector  $(e_1^i, e_2^i, \dots, e_l^i)$ . The sequence  $(\dots, \underline{e}(P^{-1}), \underline{e}(P^0), \underline{e}(P^1), \dots)$ , denoted by  $\underline{e}(P^\bullet)$ , is called the projective dimension sequence of  $P^\bullet$ . Put  $\underline{d}(\underline{e}) = (\underline{d}_i)$ , where  $\underline{d}_i = \underline{\dim} P^i$ . In the similar way as in [JSZ], for a fixed projective dimension sequence  $\underline{e} = (\dots, \underline{e}_i, \dots)$ , we define  $\mathcal{P}^b(A, \underline{e})$  to be the locally closed subset of  $\mathcal{C}^b(A, \underline{d}(\underline{e}))$  consisting of  $(x_i, \partial_i)_i$  with  $(\mathbb{C}^{\underline{d}_i}, x_i)$  isomorphic to  $P^i$  for any  $i \in \mathbb{Z}$ . The action of the algebraic group  $G_{\underline{d}(\underline{e})} := \prod_{i \in \mathbb{Z}} G_{\underline{d}_i}$  on  $\mathcal{C}^b(A, \underline{d}(\underline{e}))$  induces an action on  $\mathcal{P}^b(A, \underline{e})$ .

**2.4.** In this subsection we consider the topological structures which are endowed with  $\mathcal{C}^b(A)$  and  $\mathcal{P}^b(A)$ .

Let  $K_0(\mathcal{D}^b(A))$ , or simply by  $K_0$ , be the Grothendieck group of the derived category  $\mathcal{D}^b(A)$ , and  $\underline{\dim} : \mathcal{D}^b(A) \rightarrow K_0(\mathcal{D}^b(A))$  the canonical surjection. It induces a canonical surjection from the abelian group of dimension vector sequences to  $K_0$ , we still denote it by  $\underline{\dim}$ . We have known that the set  $\mathcal{C}^b(A, \underline{d})$  of all complex of fixed dimension vector sequence  $\underline{d}$  in  $\mathcal{C}^b(A)$  is an affine variety. For any  $\underline{d}_1, \underline{d}_2 \in \underline{\dim}^{-1}(\underline{d})$ , we write  $\underline{d}_1 \leq \underline{d}_2$ , if there exists a complex  $M^\bullet(\underline{d}_1, \underline{d}_2)$  in  $\mathcal{C}^b(A, \underline{d}_2 - \underline{d}_1)$  which is a direct sum of shifted copies of complexes of the form

$$\dots \longrightarrow 0 \longrightarrow S \xrightarrow{1} S \longrightarrow 0 \longrightarrow \dots$$

where  $S$  is a simple  $A$ -module. This defines a partial order on  $\underline{\dim}^{-1}(\mathbf{d})$ . Fix the set

$$\{M^\bullet(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2) \mid \underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2 \in \underline{\dim}^{-1}(\mathbf{d}), M^\bullet(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2) \oplus M^\bullet(\underline{\mathbf{d}}_2, \underline{\mathbf{d}}_3) = M^\bullet(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_3)\}.$$

We have a morphism of varieties :

$$T_{\underline{\mathbf{d}}_1 \underline{\mathbf{d}}_2} : \mathcal{C}^b(A, \underline{\mathbf{d}}_1) \rightarrow \mathcal{C}^b(A, \underline{\mathbf{d}}_2)$$

mapping a complex  $X^\bullet$  to  $X^\bullet \oplus M^\bullet(\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2)$ . Then we obtain a direct system  $\{(\mathcal{C}^b(A, \underline{\mathbf{d}}), T_{\underline{\mathbf{d}} \underline{\mathbf{d}}'}) \mid \underline{\mathbf{d}}, \underline{\mathbf{d}}' \in \underline{\dim}^{-1}(\mathbf{d})\}$  and define

$$\mathcal{C}^b(A, \mathbf{d}) = \varinjlim_{\underline{\mathbf{d}} \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{C}^b(A, \underline{\mathbf{d}})$$

for  $\mathbf{d} \in K_0$ . We have a canonical morphism  $T_{\underline{\mathbf{d}}} : \mathcal{C}^b(A, \underline{\mathbf{d}}) \rightarrow \mathcal{C}^b(A, \mathbf{d})$  for any  $\underline{\mathbf{d}} \in \underline{\dim}^{-1}(\mathbf{d})$ . A subset  $U$  is open in  $\mathcal{C}^b(A, \mathbf{d})$  if and only if  $T_{\underline{\mathbf{d}}}^{-1}(U)$  is open in  $\mathcal{C}^b(A, \underline{\mathbf{d}})$  for any  $\underline{\mathbf{d}} \in \underline{\dim}^{-1}(\mathbf{d})$ .

Moreover, we also define the quotient space  $\mathcal{Q}^b(A, \mathbf{d}) = \mathcal{C}^b(A, \mathbf{d}) / \sim$ , where  $x \sim y$  if and only if the corresponding complexes  $M(x)^\bullet$  and  $M(y)^\bullet$  are quasi-isomorphic to each other, i.e., they are isomorphic in  $\mathcal{D}^b(A)$ . The topology of  $\mathcal{Q}^b(A, \mathbf{d})$  is quotient topology, i.e., let  $\pi : \mathcal{C}^b(A, \mathbf{d}) \rightarrow \mathcal{Q}^b(A, \mathbf{d})$  be the canonical surjection,  $U$  is an open (closed) set of  $\mathcal{Q}^b(A, \mathbf{d})$  if and only if  $\pi^{-1}(U)$  is an open (closed) set of  $\mathcal{C}^b(A, \mathbf{d})$ . Let  $S$  be a subset of  $\mathcal{C}^b(A, \mathbf{d})$ . Its orbit space is defined as  $\mathcal{O}(S) = \{y \mid y \sim x \text{ for some } x \in S\}$ .

A complex  $X^\bullet = (X_i, \partial_i)_i$  is called contractible if the induced homological groups  $H_i(X^\bullet) = \ker \partial_{i+1} / \text{Im } \partial_i = 0$  for all  $i \in \mathbb{Z}$ . It is easy to see that any contractible projective complex is isomorphic to a direct sum of shifted copies of complexes of the form

$$\dots \longrightarrow 0 \xrightarrow{0} P \xrightarrow{f} P \xrightarrow{0} 0 \longrightarrow \dots$$

with  $P$  is a projective  $A$ -module and  $f$  is an automorphism. We call an element  $x \in \mathcal{P}^b(A, \underline{\mathbf{e}})$  contractible if the corresponding projective complex is contractible. Let  $\underline{\mathbf{e}}_1$  and  $\underline{\mathbf{e}}_2$  be projective dimension sequence such that  $\underline{\mathbf{d}}(\underline{\mathbf{e}}_1)$  and  $\underline{\mathbf{d}}(\underline{\mathbf{e}}_2)$  are in  $\underline{\dim}^{-1}(\mathbf{d})$ . We write  $\underline{\mathbf{e}}_1 \leq \underline{\mathbf{e}}_2$  if there exists a contractible projective complex  $P^\bullet(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2) \in \mathcal{P}^b(A, \underline{\mathbf{e}}_2 - \underline{\mathbf{e}}_1)$ . We fix the set

$$\{P^\bullet(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2) \mid P^\bullet(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2) \oplus P^\bullet(\underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3) = P^\bullet(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_3), \underline{\mathbf{d}}(\underline{\mathbf{e}}_1), \underline{\mathbf{d}}(\underline{\mathbf{e}}_2), \underline{\mathbf{d}}(\underline{\mathbf{e}}_3) \in \underline{\dim}^{-1}(\mathbf{d})\}$$

of contractible projective complexes. Then we have a canonical morphism of varieties

$$t_{\underline{\mathbf{e}}_1 \underline{\mathbf{e}}_2} : \mathcal{P}^b(A, \underline{\mathbf{e}}_1) \rightarrow \mathcal{P}^b(A, \underline{\mathbf{e}}_2)$$

mapping  $X^\bullet$  to  $X^\bullet \oplus P^\bullet(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2)$ . Hence, we can define

$$\mathcal{P}^b(A, \mathbf{d}) = \varinjlim_{\underline{\mathbf{d}}(\underline{\mathbf{e}}) \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{P}^b(A, \underline{\mathbf{e}})$$

for any  $\mathbf{d} \in K_0$ . By definition, there are canonical morphisms  $t_{\underline{\mathbf{e}}} : \mathcal{P}^b(A, \underline{\mathbf{e}}) \rightarrow \mathcal{P}^b(A, \mathbf{d})$ . We have the quotient space

$$\mathcal{Q}\mathcal{P}^b(A, \mathbf{d}) = \mathcal{P}^b(A, \mathbf{d}) / \sim,$$

where  $x \sim y$  in  $\mathcal{P}^b(A, \mathbf{d})$  if and only if the corresponding projective complexes  $P(x)^\bullet$  and  $P(y)^\bullet$  are quasi-isomorphic, i.e., they are isomorphic in  $\mathcal{D}^b(A)$ . The topology



for  $\mathcal{QP}^b(A, \mathbf{d})$  is the quotient topology from  $\mathcal{P}^b(A, \mathbf{d})$ . For any  $x \in \mathcal{P}^b(A, \mathbf{d})$ , then the orbit is

$$\mathcal{O}(x) = \{y \in \mathcal{P}^b(A, \mathbf{d}) | y \sim x\}.$$

For  $\mathbf{e}_1 \leq \mathbf{e}_2$ , there exist a natural morphism of varieties  $f_{\mathbf{e}_1 \mathbf{e}_2} : G_{\mathbf{d}(\mathbf{e}_1)} \rightarrow G_{\mathbf{d}(\mathbf{e}_2)}$  mapping  $g = (g_i)$  to  $\begin{pmatrix} g_i & 0 \\ 0 & 1 \end{pmatrix}$ . Then we define

$$G_{\mathbf{d}} = \varinjlim_{\mathbf{d}(\mathbf{e}) \in \underline{\dim}^{-1}(\mathbf{d})} G_{\mathbf{d}(\mathbf{e})}.$$

By definition, there are canonical morphisms  $f_{\mathbf{e}} : G_{\mathbf{e}} \rightarrow G_{\mathbf{d}}$ . The action of the algebraic group  $G_{\mathbf{d}(\mathbf{e})}$  on  $\mathcal{P}^b(A, \mathbf{e})$  naturally induces an action of  $G_{\mathbf{d}}$  on  $\mathcal{P}^b(A, \mathbf{d})$ . Let  $g_{\mathbf{e}} \in G_{\mathbf{e}}$  and  $x_{\mathbf{e}'} \in \mathcal{P}^b(A, \mathbf{e}')$ . Then there exists  $\mathbf{e}''$  such that  $\mathbf{e} \leq \mathbf{e}''$  and  $\mathbf{e}' \leq \mathbf{e}''$ . We define

$$f_{\mathbf{e}}(g_{\mathbf{e}}) \cdot t_{\mathbf{e}'}(x_{\mathbf{e}'})) = f_{\mathbf{e}''}(f_{\mathbf{e} \mathbf{e}''}(g_{\mathbf{e}}) \cdot t_{\mathbf{e}' \mathbf{e}''}(x_{\mathbf{e}'})).$$

It is well-defined. We denote by  $\mathcal{QP}^b(A, \mathbf{e})$  and  $\mathcal{Q}_1 \mathcal{P}^b(A, \mathbf{d})$  the orbit spaces of  $\mathcal{P}^b(A, \mathbf{e})$  and  $\mathcal{P}^b(A, \mathbf{d})$  under the actions of  $G_{\mathbf{d}(\mathbf{e})}$  and  $G_{\mathbf{d}}$ , respectively. We have the follow result.

**Proposition 2.1.** With the above notations, we have

$$\mathcal{QP}^b(A, \mathbf{d}) = \mathcal{Q}_1 \mathcal{P}^b(A, \mathbf{d}) = \varinjlim_{\mathbf{d}(\mathbf{e}) \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{QP}^b(A, \mathbf{e}).$$

The proposition is the direct corollary of the following three lemmas (see [BD] or [JSZ]).

**Lemma 2.2.** In  $\mathcal{P}^b(A)$ , any projective complex can be uniquely decomposed (up to isomorphism) into the direct sum of a minimal projective complex and a contractible projective complex.

**Lemma 2.3.** Let  $f^\bullet : P^\bullet \rightarrow Q^\bullet$  be a morphism between two minimal projective complex in  $\mathcal{P}^b(A)$ . Then  $f^\bullet$  is a quasi-isomorphism if and only if  $f^\bullet$  is an isomorphism.

**Lemma 2.4.** Let  $f^\bullet : P^\bullet \rightarrow Q^\bullet$  be a morphism in  $\mathcal{P}^b(A)$  and  $\mathbf{e}(P^\bullet) = \mathbf{e}(Q^\bullet)$ . Then  $f^\bullet$  is a quasi-isomorphism if and only if  $f^\bullet$  is an isomorphism.

2.5. The aim of this subsection is to build the connection between  $\mathcal{Q}^b(A, \mathbf{d})$  and  $\mathcal{QP}^b(A, \mathbf{d})$ .

**Lemma 2.5.** Suppose the following diagram is a pullback of  $A$ -module, and  $g_1$  is surjective,

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ \downarrow f_2 & & \downarrow g_1 \\ Z & \xrightarrow{g_2} & W \end{array}$$

then we have the following properties hold:

- (1)  $\text{Ker } f_1 \cong \text{Ker } g_2$ ;
- (2)  $\frac{Y}{\text{Im } f_1} \cong \frac{W}{\text{Im } g_2}$ ;
- (3) there exists the exact sequence:  $0 \rightarrow X \rightarrow Y \oplus Z \rightarrow W \rightarrow 0$ .

*Proof.* The first and third statement just follow the definition of pullback, refer to [ARS]. For the second statement, we use the first statement again to get an isomorphism:  $\text{Ker} f_2 \cong \text{Ker} g_1$ . By this, the conclusion follows.  $\square$

For any dimension vector sequence  $\underline{d} = (d_i)_{i \in \mathbb{Z}}$ , we construct for any complex  $M^\bullet$  in  $\mathcal{C}^b(A, \underline{d})$  a projective complex  $F^\bullet$  such that  $F^\bullet$  is quasi-isomorphic to  $M^\bullet$ . Assume  $\dim_{\mathbb{C}} A = n$  and  $\text{gl.dim.} A = m$ .

Let  $M^\bullet$  be a complex with the following form:

$$(2.3) \quad 0 \longrightarrow M_1 \xrightarrow{\partial_2} \dots \xrightarrow{\partial_{r-1}} M_{r-1} \xrightarrow{\partial_r} M_r \longrightarrow 0$$

which  $\dim_{\mathbb{C}} M_i = d_i$  for any  $i \in \mathbb{Z}$ . Here, if  $\underline{d}_i = (d_i^j)_{j \in Q_0}$ , then  $d_i = \sum_{j \in Q_0} d_i^j$ .

Since  $M_r$  has dimension  $d_r$ , we have the surjective map:  $\pi_r : A^{d_r} \longrightarrow M_r$ . Along the differential  $\partial_r$  and the above  $\pi_r$ , we form the pullback  $X_{r-1}$ :

$$(2.4) \quad \begin{array}{ccccccc} \dots & \longrightarrow & M_{r-2} & \longrightarrow & X_{r-1} & \xrightarrow{\hat{\partial}_r} & A^{d_r} \longrightarrow 0 \\ & & \text{id} \downarrow & & \hat{\pi}_r \downarrow & & \pi_r \downarrow \\ \dots & \longrightarrow & M_{r-2} & \longrightarrow & M_{r-1} & \xrightarrow{\partial_r} & M_r \longrightarrow 0 \end{array}$$

Depending on Lemma 2.6, we have the following exact sequence:

$$0 \longrightarrow X_{r-1} \longrightarrow A^{d_r} \oplus M_{r-1} \longrightarrow M_r \longrightarrow 0$$

The dimension of  $X_{r-1}$ , denoted by  $l_{r-1}$ , is  $d_{r-1} + d_r(n-1)$ . Similarly, we have the surjective map:  $\tilde{\pi}_r : A^{l_{r-1}} \longrightarrow X_{r-1}$ , and  $\hat{\pi}_r$  is also surjective by the lemma. So we can also form the pullback  $X_{r-2}$  as showed in the following diagram:

$$(2.5) \quad \begin{array}{ccccccc} \dots & \longrightarrow & M_{r-3} & \longrightarrow & X_{r-2} & \xrightarrow{\hat{\partial}_{r-1}} & A^{l_{r-1}} \longrightarrow A^{l_r} \longrightarrow 0 \\ & & \parallel & & \downarrow \hat{\pi}_{r-1} & & \downarrow \tilde{\pi}_r \\ \dots & \longrightarrow & M_{r-3} & \longrightarrow & M_{r-2} & \longrightarrow & X_{r-1} \xrightarrow{\hat{\partial}_r} A^{d_r} \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \hat{\pi}_r \\ \dots & \longrightarrow & M_{r-3} & \xrightarrow{\partial_{r-2}} & M_{r-2} & \xrightarrow{\partial_{r-1}} & M_{r-1} \xrightarrow{\partial_r} M_r \longrightarrow 0 \end{array}$$

Inductively, we get a complex of ‘almost’ free  $A$ -module  $F^\bullet$  as follows:

$$(2.6) \quad 0 \longrightarrow P \longrightarrow \dots \longrightarrow A^{l_1} \xrightarrow{\hat{\partial}_2 \tilde{\pi}_2} \dots \longrightarrow A^{l_r} \longrightarrow 0$$

where  $l_i + l_{i-1} = nl_i + d_{i-1}$  for  $i = 2, \dots, r$ , in particular,  $l_r = d_r$  and  $P$  is a projective module of dimension  $nl_{2-m} - l_{2-m}$ . Every term of this complex is a free  $A$ -module except the first term. By the construction, there exists a projective dimension sequence  $\underline{e}$  only depending on the choice of dimension vector sequence  $\underline{d}$  such that  $F^\bullet \in \mathcal{P}^b(A, \underline{e})$ . Moreover, This complex is quasi-isomorphism to  $M^\bullet$ . First,

$$H_r(F^\bullet) = \frac{A^{l_r}}{\text{Im} \hat{\partial}_r \tilde{\pi}_r} = \frac{A^{l_r}}{\text{Im} \hat{\partial}_r} \cong \frac{M_r}{\text{Im} \partial_r}$$

This follows from that  $\tilde{\pi}_r$  is surjective and the above lemma. In general, for  $i < r$ ,

$$H_i(F^\bullet) = \frac{\text{Ker} \hat{\partial}_{i+1} \tilde{\pi}_{i+1}}{\text{Im} \hat{\partial}_i \tilde{\pi}_i} = \frac{\text{Ker} \hat{\partial}_{i+1} \tilde{\pi}_{i+1}}{\text{Im} \hat{\partial}_i} = \frac{(\hat{\pi}_{i+1} \tilde{\pi}_{i+1})^{-1} (\text{Ker} \partial_{i+1})}{(\hat{\pi}_{i+1} \tilde{\pi}_{i+1})^{-1} (\text{Im} \partial_i)} = H_i(M^\bullet).$$

In fact, we can construct a new complex from any place of a complex and two complexes are quasi-isomorphic to each other as follows. Given a dimension vector sequence  $\underline{\mathbf{d}}(i)$  for some  $i \in \mathbb{N}$ , let  $M^\bullet \in \mathcal{C}^b(A, \underline{\mathbf{d}}(i))$ . Then we obtain a complex  $X^\bullet \in \mathcal{C}^b(A, \underline{\mathbf{d}}(i-1))$  for some dimension vector sequence  $\underline{\mathbf{d}}(i-1)$  by the commutative diagram

$$(2.7) \quad \begin{array}{ccccccc} X^\bullet : & \cdots & \longrightarrow & X_{i-1} & \longrightarrow & A^{d_i} & \longrightarrow & M_{i+1} & \longrightarrow & \cdots \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M^\bullet : & \cdots & \longrightarrow & M_{i-1} & \longrightarrow & M_i & \longrightarrow & M_{i+1} & \longrightarrow & \cdots \end{array}$$

where  $d_i = \dim_{\mathbb{C}} M_i$  and  $X_{i-1}$  is the pullback. In this way, we obtain a map

$$f_i : \mathcal{C}^b(A, \underline{\mathbf{d}}(i)) \rightarrow \mathcal{C}^b(A, \underline{\mathbf{d}}(i-1))$$

such that  $f_i(M^\bullet)$  is quasi-isomorphic to  $M^\bullet$  for any  $M^\bullet \in \mathcal{C}^b(A, \underline{\mathbf{d}}(i))$ .

**Proposition 2.6.** The above construction induces a map  $f_{\underline{\mathbf{d}}} : \mathcal{C}^b(A, \underline{\mathbf{d}}) \rightarrow \mathcal{P}^b(A, \underline{\mathbf{e}})$  for some projective dimension sequence  $\underline{\mathbf{e}}$  such that there exists a finite stratification  $\mathcal{C}^b(A, \underline{\mathbf{d}}) = \bigsqcup_i \mathcal{O}_i$  such that all  $\mathcal{O}_i$  are constructible and  $f_{\underline{\mathbf{d}}} |_{\mathcal{O}_i}$  is a morphism of varieties.

The map satisfies the property in Proposition 2.6 is called the constructible map which has been considered in [Xu] or [Pa].

Now we consider the geometrization of Kernel and cokernel of the homomorphisms of modules. First of all, we consider the kernel and cokernel of a linear map.

**Lemma 2.7.** For any  $d_1, d_2 \in \mathbb{N}$ , there exist constructible maps

$$\mathcal{K} : \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \rightarrow \bigcup_{d \leq d_1} \text{Inj}(k^d, k^{d_1})$$

and

$$\mathcal{C} : \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2}) \rightarrow \bigcup_{d' \leq d_2} \text{Surj}(k^d, k^{d_1})$$

such that for any  $f \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_1}, \mathbb{C}^{d_2})$ , there exists a long exact sequence

$$0 \longrightarrow \mathbb{C}^d \xrightarrow{\mathcal{K}(f)} \mathbb{C}^{d_1} \xrightarrow{f} \mathbb{C}^{d_2} \xrightarrow{\mathcal{C}(f)} \mathbb{C}^{d'} \longrightarrow 0$$

where  $d$  is the rank of  $f$  and  $d' = d_2 + d - d_1$ .

It can be easily proved by making a finite partition of  $M_{d_2 \times d_1}(\mathbb{C})$ . As an analogy of this lemma for homomorphisms of modules, we have

**Lemma 2.8.** For any two dimension vectors  $\underline{\mathbf{d}}_1, \underline{\mathbf{d}}_2$ , let  $(\mathbb{C}^{\underline{\mathbf{d}}_1}, x_1)$  and  $(\mathbb{C}^{\underline{\mathbf{d}}_2}, x_2)$  be two  $A$ -modules. Then there exist constructible maps

$$(\mathcal{K}_1, \mathcal{K}_2) : \text{Hom}_A((\mathbb{C}^{\underline{\mathbf{d}}_1}, x_1), (\mathbb{C}^{\underline{\mathbf{d}}_2}, x_2)) \rightarrow \bigcup_{\underline{\mathbf{d}}; \underline{\mathbf{d}} \leq \underline{\mathbf{d}}_1} \mathbb{E}_{\underline{\mathbf{d}}}(A) \times \text{Hom}_{\mathbb{C}}(\mathbb{C}^{\underline{\mathbf{d}}}, \mathbb{C}^{\underline{\mathbf{d}}_1})$$

and

$$(\mathcal{C}_1, \mathcal{C}_2) : \text{Hom}_A((\mathbb{C}^{\underline{d}_1}, x_1), (\mathbb{C}^{\underline{d}_2}, x_2)) \rightarrow \bigcup_{\underline{d}' : \underline{d}' \leq \underline{d}_2} \mathbb{E}_{\underline{d}'}(A) \times \text{Hom}_{\mathbb{C}}(\mathbb{C}^{\underline{d}_2}, \mathbb{C}^{\underline{d}'})$$

such that for any  $f \in \text{Hom}_A((\mathbb{C}^{\underline{d}_1}, x_1), (\mathbb{C}^{\underline{d}_2}, x_2))$ , there exists a long exact sequence

$$0 \longrightarrow (\mathbb{C}^{\underline{d}}, \mathcal{K}_1(f)) \xrightarrow{\mathcal{K}_2(f)} (\mathbb{C}^{\underline{d}_1}, x_1) \xrightarrow{f} (\mathbb{C}^{\underline{d}_2}, x_2) \xrightarrow{\mathcal{C}_2(f)} (\mathbb{C}^{\underline{d}'}, \mathcal{C}_2(f)) \longrightarrow 0.$$

Now we come to prove our proposition.

*Proof.* Without loss of generality, we assume  $\underline{d} = (\underline{d}_1, \dots, \underline{d}_r)$  for some  $r \in \mathbb{N}$ . We set  $\underline{d}(r) := \underline{d}$ . By Lemma 2.8, the above construction of ‘almost’ free resolution induces a chain of constructible map:

$$\mathcal{C}^b(A, \underline{d}(r)) \xrightarrow{f_r} \mathcal{C}^b(A, \underline{d}(r-1)) \xrightarrow{f_{r-1}} \dots \longrightarrow \mathcal{C}^b(A, \underline{d}(-m))$$

where  $\underline{d}(r-1) = (\underline{d}_1^{r-1}, \dots, \underline{d}_r^{r-1})$  satisfies  $\underline{d}_i^{r-1} = \underline{d}_i$  for  $i < r-1$  and  $\underline{d}_r^{r-1} = \underline{\dim} A^{d_r}$  and  $\underline{d}_{r-1}^{r-1} = \underline{\dim} A^{d_r} + \underline{d}_{r-1} - \underline{d}_r$ . By the construction,  $\underline{d}(-m)$  is determined by  $\underline{d}$  and  $\underline{\dim} A$ . Let  $f_{\underline{d}} = f_r \cdots f_{-m}$ . Note that the composition of constructible maps is constructible. Then we deduce a constructible map from  $\mathcal{C}^b(A, \underline{d}(1))$  to  $\mathcal{C}^b(A, \underline{d}(-m))$  satisfying the image of any complex under  $f_{\underline{d}}$  is a almost free projective complex. We complete the proof of Proposition 2.6.  $\square$

2.6. We have the following results.

**Theorem A** *For any  $\underline{d} \in K_0(\mathcal{D}^b(A))$ , let  $\phi_A : \mathcal{P}^b(A, \underline{d}) \rightarrow \mathcal{C}^b(A, \underline{d})$  be the natural embedding. Then there exists a map  $\phi'_A : \mathcal{C}^b(A, \underline{d}) \rightarrow \mathcal{P}^b(A, \underline{d})$  such that for any  $\underline{d} \in \underline{\dim}^{-1}(\underline{d})$ ,  $\phi' \mid_{\mathcal{C}^b(A, \underline{d})}$  is a constructible map and the quotient maps of  $\phi_A$  and  $\phi'_A$  between  $\mathcal{Q}^b(A, \underline{d})$  and  $\mathcal{Q}\mathcal{P}^b(A, \underline{d})$  are inverse to each other.*

*Proof.* By Proposition 2.6, there is a map  $\phi'_A$  such that  $\phi'_A \mid_{\mathcal{C}^b(A, \underline{d})} = f_{\underline{d}}$ . For any  $X^\bullet \in \mathcal{C}^b(A, \underline{d})$ , by definition,  $X^\bullet$  is quasi-isomorphic to  $\phi_A \phi'_A(X^\bullet)$ . This proves the theorem.  $\square$

Let us recall some results on Morita theory of derived categories ([Rick1] and [Rick2]). First, a tilting complex  $T$  over  $A$  is an object in  $\mathcal{K}^b(P)$  which satisfies the following conditions:

1. For any  $i \neq 0$ ,  $\text{Hom}_{\mathcal{D}^b(A)}(T, T[i]) = 0$ ;
2. The category  $\text{add}(T)$  generates  $\mathcal{K}^b(P)$  as a triangulated category.

For convenience, we shall consider the right  $A$ -module temporarily and  $A$ - $B$ -bimodule means module for  $A^{op} \otimes_{\mathbb{C}} B$  in the following part of this section. J. Rickard proved the following result.

**Rickard’s Theorem** *For any two finite dimensional basic  $\mathbb{C}$ -algebras  $A$  and  $B$ , they are derived equivalent if and only if there is a tilting complex  $T$  over  $A$  such that the endomorphism ring  $\text{End}_{\mathcal{D}^b(A)}(T)^{op}$  is isomorphic to  $B$ .*

Moreover, the method of Rickard implies the further results. If  $A$  and  $B$  are derived equivalent by a functor  $F$ , then the functor  $F$  induces a derived equivalence between  $\mathcal{D}^b(A^{op} \otimes_{\mathbb{C}} A)$  and  $\mathcal{D}^b(B^{op} \otimes_{\mathbb{C}} A)$ . The image of  $A$  as  $A^{op} \otimes_{\mathbb{C}} A$  module under this functor is a complex, say  $\Delta$ , in  $\mathcal{D}^b(B^{op} \otimes_{\mathbb{C}} A)$ . The functor  $F$  induces

also a derived equivalence between  $\mathcal{D}^b(B^{op} \otimes_{\mathbb{C}} B)$  and  $\mathcal{D}^b(A^{op} \otimes_{\mathbb{C}} B)$ . The image of  $B$  as  $B^{op} \otimes_{\mathbb{C}} B$  module under this functor is a complex, say  $\Theta$ , in  $\mathcal{D}^b(A^{op} \otimes_{\mathbb{C}} B)$ . Then

$$- \otimes_B^L \Delta : \mathcal{D}^b(B) \longrightarrow \mathcal{D}^b(A)$$

is an equivalence of triangulated categories with two quasi-inverses  $\mathrm{RHom}_{\mathcal{D}^b(A)}(X, -) : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$  and  $- \otimes_B^L \Theta : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(B)$ .

**Theorem B** *Let  $F : \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(B)$  be a functor of derived equivalence with the induced isomorphism  $F_0 : K_0(\mathcal{D}^b(A)) \rightarrow K_0(\mathcal{D}^b(B))$ . Then for any  $\mathbf{d}_A \in K_0(\mathcal{D}^b(A))$ , there exist maps  $\mathcal{F}_1 : \mathcal{C}^b(A, \mathbf{d}_A) \rightarrow \mathcal{C}^b(B, F_0(\mathbf{d}_A))$  and  $\mathcal{F}_2 : \mathcal{C}^b(B, F_0(\mathbf{d}_A)) \rightarrow \mathcal{C}^b(A, \mathbf{d}_A)$  such that for any  $\underline{\mathbf{d}}_A \in \underline{\dim}^{-1}(\mathbf{d}_A)$  and  $\underline{\mathbf{d}}_B \in \underline{\dim}^{-1}(F_0(\mathbf{d}_A))$ ,  $\mathcal{F}_1|_{\mathcal{C}^b(A, \underline{\mathbf{d}}_A)}$  and  $\mathcal{F}_2|_{\mathcal{C}^b(B, \underline{\mathbf{d}}_B)}$  are constructible maps and the quotient maps of  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are inverse to each other.*

*Proof.* We consider  $\Delta$  as a  $B^{op} \otimes_{\mathbb{C}} A$ -projective complex:

$$(2.8) \quad \dots \longrightarrow \Delta_j \xrightarrow{\partial_j^\Delta} \Delta_{j+1} \longrightarrow \dots$$

where  $\Delta_j$  is  $B^{op} \otimes_{\mathbb{C}} A$ -projective module. For any  $X^\bullet \in \mathcal{C}^b(B, \underline{\mathbf{d}}_B)$ , let  $f_{\underline{\mathbf{d}}_B}(X^\bullet)$  is the corresponding  $B$ -projective complex with the following form:

$$(2.9) \quad \dots \longrightarrow P^i \xrightarrow{\partial_i^P} P^{i+1} \longrightarrow \dots$$

Here,  $f_{\underline{\mathbf{d}}_B}$  is the constructible map from  $\mathcal{C}^b(B, \underline{\mathbf{d}}_B)$  to  $\mathcal{P}^b(B, \underline{\mathbf{e}}_1)$  for some projective dimension sequence  $\underline{\mathbf{e}}_1$ . Define the following complex (see [Rick2]):

$$(2.10) \quad \dots \longrightarrow \prod_{i+j=n} P^i \otimes_B \Delta_j \xrightarrow{d_n} \prod_{i+j=n+1} P^i \otimes_B \Delta_j \longrightarrow \dots$$

where  $d_n = \partial_i^P \otimes id + (-1)^i id \otimes \partial_i^\Delta$  and  $\prod_{i+j=n} P^i \otimes_B \Delta_j$  is  $A$ -projective module. We set  $\underline{\mathbf{e}}_2$  to be its  $A$ -projective dimension sequence. This defines a map  $f_{\underline{\mathbf{e}}_1 \underline{\mathbf{e}}_2} : \mathcal{P}^b(B, \underline{\mathbf{e}}_1) \longrightarrow \mathcal{P}^b(A, \underline{\mathbf{e}}_2)$ . By the above construction, it is clearly a constructible map. In the same way, we can define the constructible map

$$g_{\underline{\mathbf{e}}_2 \underline{\mathbf{e}}_3} : \mathcal{P}^b(A, \underline{\mathbf{e}}_2) \longrightarrow \mathcal{P}^b(B, \underline{\mathbf{e}}_3)$$

by  $g(P^\bullet) = P^\bullet \otimes_A \Theta$ , which is induced by  $G = - \otimes_A \Theta$ . The constructible maps  $\phi_B g_{\underline{\mathbf{e}}_2 \underline{\mathbf{e}}_3}$  and  $f_{\underline{\mathbf{e}}_1 \underline{\mathbf{e}}_2} f_{\underline{\mathbf{d}}_B}$  induce the maps  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Because  $F$  and  $G$  are quasi-inverse to each other,  $\phi_B g_{\underline{\mathbf{e}}_2 \underline{\mathbf{e}}_3} f_{\underline{\mathbf{e}}_1 \underline{\mathbf{e}}_2}(f_{\underline{\mathbf{d}}_B}(X^\bullet))$  is isomorphic to  $X^\bullet$  in  $\mathcal{D}^b(B)$ . This proves the theorem.  $\square$

### 3. CONSTRUCTIBLE FUNCTIONS ON TOPOLOGICAL SPACES ATTACHED TO DERIVED CATEGORIES

**3.1. Degenerations in derived categories.** We rewrite the definition of the degeneration in [JSZ] for our situation. For any  $X$  and  $Y$  in  $\mathcal{K}^b(\mathcal{P}^b(A))$  we denote by  $X \leq_{\Delta} Y$  if there is a distinguished triangle

$$Y \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow Y[1]$$

for some  $Z \in \mathcal{K}^b(\mathcal{P}^b(A))$ . On the topological side, we denote by  $X \leq_{top} Y$  if  $Y \in \overline{\mathcal{O}(X)}$  in  $\mathcal{P}^b(A, \mathbf{d})$  where  $\overline{\mathcal{O}(X)}$  is the closure of the orbit  $\mathcal{O}(X)$  of  $X$  in

$\mathcal{P}^b(A, \mathbf{d})$  under the action of  $G_{\mathbf{d}}$ . In order to avoid confusion, we use the following different notation:  $X \leq_{top}^* Y$  if and only if  $Y \in \overline{G_{\mathbf{e}} X} = \overline{\mathcal{O}_{\mathbf{e}}(X)}$  in  $\mathcal{P}^b(A, \mathbf{e})$  for a fixed projective dimension sequence  $\mathbf{e}$  where  $\mathcal{O}_{\mathbf{e}}(X)$  is the orbit of  $X$  in  $\mathcal{P}^b(A, \mathbf{e})$  under the action of  $G_{\mathbf{e}}$  and  $\overline{\mathcal{O}(X)}$  is the closure of  $\mathcal{O}(X)$  in  $\mathcal{P}^b(A, \mathbf{e})$ . Then Theorem 1 and Theorem 2 in [JSZ] implies the following result.

**Theorem 3.1.** *For any  $X$  and  $Y$  in  $\mathcal{K}^b(\mathcal{P}(A))$ , then  $X \leq_{\Delta} Y$  if and only if  $X \leq_{top} Y$ .*

For any projective dimension sequence  $\mathbf{e}'$  such that  $\mathbf{e} \leq \mathbf{e}'$ , we set  $X_{\mathbf{e}'} := t_{\mathbf{e}\mathbf{e}'}(X)$ . Then we have

$$\mathcal{O}(X) = \varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'}), \quad \overline{\mathcal{O}(X)} = \varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} \overline{\mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'})}.$$

**Proposition 3.2.** Let  $X \in \mathcal{P}^b(A, \mathbf{e})$  where  $\mathbf{e} \in \underline{\dim}^{-1}(\mathbf{d})$ . Then  $\mathcal{O}(X)$  is a locally closed subset of  $\mathcal{P}^b(A, \mathbf{d})$ , i.e., the intersection of a closed subset with an open subset.

*Proof.* For any  $\mathbf{e}' \geq \mathbf{e}$ , consider the morphism  $t_{\mathbf{e}\mathbf{e}'} : \mathcal{P}^b(A, \mathbf{e}) \rightarrow \mathcal{P}^b(A, \mathbf{e}')$  sending  $M^{\bullet}$  to  $M^{\bullet} \oplus P^{\bullet}(\mathbf{e}, \mathbf{e}')$  where  $P^{\bullet}(\mathbf{e}, \mathbf{e}')$  is the direct sum of the complexes with only two nonzero term formed as  $P \xrightarrow{f} P$ . For any  $\lambda \in \mathbb{C}^*$ , we denote by  $P_{\lambda}^{\bullet}(\mathbf{e}, \mathbf{e}')$  the complex isomorphic to  $P^{\bullet}(\mathbf{e}, \mathbf{e}')$  obtained by substituting  $P \xrightarrow{\lambda f} P$  for any direct summand  $P \xrightarrow{f} P$  of  $P^{\bullet}(\mathbf{e}, \mathbf{e}')$ . The orbit  $\mathcal{O}_{\mathbf{e}}(X)$  of  $X$  in  $\mathcal{P}^b(A, \mathbf{e})$  is locally closed. By definition,  $\mathcal{O}_{\mathbf{e}}(X) = \overline{\mathcal{O}_{\mathbf{e}}(X)} \cap U_{\mathbf{e}}$  for some open subset  $U_{\mathbf{e}}$  in  $\mathcal{P}^b(A, \mathbf{e})$ . It is clear that the set  $\mathcal{S}(\mathbf{e}, \mathbf{e}') := \{X^{\bullet} \oplus P_{\lambda}^{\bullet}(\mathbf{e}, \mathbf{e}') \mid X^{\bullet} \in U_{\mathbf{e}}, \lambda \in \mathbb{C}^*\}$  is an open subset of  $\mathcal{P}^b(A, \mathbf{e}')$ . Set  $U_{\mathbf{e}'} := \bigsqcup_{g \in G_{\mathbf{e}'}} g \cdot \mathcal{S}(\mathbf{e}, \mathbf{e}')$ . It is an open subset of  $\mathcal{P}^b(A, \mathbf{e}')$ . Then  $\mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'}) = \overline{\mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'})} \cap U_{\mathbf{e}'}$ . Then by definition,  $\mathcal{O}(X)$  is equal to

$$\varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'}) = \varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} \overline{\mathcal{O}_{\mathbf{e}'}(X_{\mathbf{e}'})} \cap U_{\mathbf{e}'} = \overline{\mathcal{O}(X)} \cap \varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} U_{\mathbf{e}'}.$$

By the definition of the fine topology on  $\mathcal{P}^b(A, \mathbf{d})$ ,  $\varinjlim_{\mathbf{d}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d})} U_{\mathbf{e}'}$  is an open subset of  $\mathcal{P}^b(A, \mathbf{d})$ . We finish the proof.  $\square$

We recall the following property which is proved in [JSZ]: A complex  $P(x)$  corresponding to  $x \in \mathcal{P}^b(A, \mathbf{e})$  is partial tilting complex if  $\text{Hom}_{\mathcal{K}^b(A)}(P(x), P(x)[1]) = 0$ . Then  $\mathcal{O}_{\mathbf{e}}(x) = \{y \in \mathcal{P}^b(A, \mathbf{e}) \mid y \sim x\}$  is open in  $\mathcal{P}^b(A, \mathbf{e})$ . Furthermore, the orbit  $\mathcal{O}(x)$  in  $\mathcal{P}^b(A, \mathbf{d})$  is also open if the corresponding complex  $P(x)$  is a partial tilting complex.

**Proposition 3.3.** Any point in  $\mathcal{Q}\mathcal{P}^b(A, \mathbf{d})$  is not closed, that is, any orbit is not closed in  $\mathcal{P}^b(A, \mathbf{d})$ .

*Proof.* Take any projective  $A$ -module  $P$  and  $t \in \mathbb{C}$ , we define the complex  $C(t) = (P_i, \partial_i)_{i \in \mathbb{Z}}$  by  $P_0 = P_1 = P$  and other  $P_i = 0$  for  $i \neq 0, 1$ ;  $\partial_0 = t$  and other  $\partial_i = 0$  for  $i \neq 0$ .

If  $t \neq 0$ , then  $C(t)$  is contractible but  $C(0)$  is not quasi-isomorphic to zero. Moreover,  $C(1) \leq_{top} C(0)$ . Similarly, For any  $x \in \mathcal{P}^b(A, \mathbf{d})$ ,  $P(x)$  is the corresponding complex,  $P(x) \oplus C(1) \leq_{top} P(x) \oplus C(0)$ , i.e. the point corresponding to  $P(x) \oplus C(0)$  is in the closure of the point corresponding to  $P(x) \oplus C(1)$  (it is quasi isomorphic to  $P(x)$ ), but not in its orbit. This shows the orbit of  $x$  is not closed.  $\square$

By this proposition, we can construct an infinite sequence of non-trivial degenerations:

$$P(x) \leq_{top} P(x) \bigoplus C(0) \leq_{top} P(x) \bigoplus C(0) \bigoplus C(0) \leq \dots$$

**3.2. The naïve Euler characteristic.** Let  $X$  be an algebraic variety over  $\mathbb{C}$ . We denote by  $M(X)$  the set of all constructible functions on algebraic variety  $X$  with values in  $\mathbb{C}$ . The set  $M(X)$  is naturally a  $\mathbb{C}$ -linear space. Let  $G$  be an algebraic group acting on  $X$ . Then we denote by  $M_G(X)$  the subspace of  $M(X)$  consisting of all  $G$ -invariant functions.

Let  $\chi$  denote Euler characteristic in compactly-supported cohomology. Let  $X$  be an algebraic variety and  $\mathcal{O}$  a constructible subset as the disjoint union of finitely many locally closed subsets  $X_i$  for  $i = 1, \dots, m$ . Define  $\chi(\mathcal{O}) = \sum_{i=1}^m \chi(X_i)$ . We note that it is well-defined. We will use the following properties:

**Proposition 3.4** ([Di],[Rie] and [Jo1]). Let  $X, Y$  be algebraic varieties over  $\mathbb{C}$ . Then

- (1) If the algebraic variety  $X$  is the disjoint union of finitely many constructible sets  $X_1, \dots, X_r$ , then

$$\chi(X) = \sum_{i=1}^r \chi(X_i).$$

- (2) If  $\varphi : X \rightarrow Y$  is a morphism with the property that all fibers have the same Euler characteristic  $\chi$ , then  $\chi(X) = \chi \cdot \chi(Y)$ . In particular, if  $\varphi$  is a locally trivial fibration in the analytic topology with fibre  $F$ , then  $\chi(X) = \chi(F) \cdot \chi(Y)$ .
- (3)  $\chi(\mathbb{C}^n) = 1$  and  $\chi(\mathbb{P}^n) = n + 1$  for all  $n \geq 0$ .

We recall the definition of *pushforward* functor from the category of algebraic varieties over  $\mathbb{C}$  to the category of  $\mathbb{Q}$ -vector spaces.

Let  $\phi : X \rightarrow Y$  be a morphism of varieties. For  $f \in M(X)$  and  $y \in Y$ , define

$$\phi_*(f)(y) = \sum_{c \in \mathbb{Q}} c \chi(f^{-1}(c) \cap \phi^{-1}(y)).$$

**Theorem 3.5** ([Di],[Jo1]). Let  $X, Y$  and  $Z$  be algebraic varieties over  $\mathbb{C}$ ,  $\phi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be morphisms of varieties, and  $f \in M(X)$ . Then  $\phi_*(f)$  is constructible,  $\phi_* : M(X) \rightarrow M(Y)$  is a  $\mathbb{Q}$ -linear map and  $(\psi \circ \phi)_* = (\psi)_* \circ (\phi)_*$  as  $\mathbb{Q}$ -linear maps from  $M(X)$  to  $M(Z)$ .

In order to deal with orbit spaces, we need to consider geometric quotients.

**Definition 3.6.** Let  $G$  be an algebraic group acting on a variety  $X$  and  $\phi : X \rightarrow Y$  be a  $G$ -invariant morphism, i.e. a morphism constant on orbits. The pair  $(Y, \phi)$  is called a geometric quotient if  $\phi$  is open and for any open subset  $U$  of  $Y$ , the associated comorphism identifies the ring  $\mathcal{O}_Y(U)$  of regular functions on  $U$  with the ring  $\mathcal{O}_X(\phi^{-1}(U))^G$  of  $G$ -invariant regular functions on  $\phi^{-1}(U)$ .

The following result due to Rosenlicht [Ro] is essential to us.

**Lemma 3.7.** Let  $X$  be a  $G$ -variety, then there exists a open and dense  $G$ -stable subset which has a geometric  $G$ -quotient.

By this Lemma, we can construct a finite stratification over  $X$ . Let  $U_1$  be an open and dense  $G$ -stable subset of  $X$  as in Lemma 3.8. Then we have a geometric  $\phi_{U_1} : U_1 \rightarrow Y_1$ . Since  $\dim_{\mathbb{C}}(X - U_1) < \dim_{\mathbb{C}} X$ , we can use the above lemma again, there exists a dense open  $G$ -stable subset  $U_2$  of  $X - U_1$  which has a geometric  $G$ -quotient  $\phi_{U_2} : U_2 \rightarrow Y_2$ . Inductively, we get a finite stratification  $X = \bigcup_{i=1}^l U_i$  where  $U_i$  is a  $G$ -invariant locally closed subset and has a geometric quotient  $\phi_{U_i} : U_i \rightarrow Y_i$  for  $i = 1, \dots, l$  with  $l \leq \dim_{\mathbb{C}} X$ . Depending on this stratification, we define the naïve Euler characteristic of the orbit space of  $X$  under the group action of  $G$  as follows:

$$\chi^{\text{na}}(X/G) := \chi^{\text{na}}([X/G](\mathbb{C})) = \chi\left(\coprod_{i=1}^l Y_i\right) = \sum_{i=1}^l \chi(\phi_{U_i}(U_i))$$

where  $[X/G]$  is the quotient stack with the set of  $\mathbb{C}$ -points  $[X/G](\mathbb{C})$  and  $[X/G](\mathbb{C})$  is pseudoisomorphic to  $\coprod_{i=1}^l Y_i$  (see [Jo1, Section 4] for the definition of pseudoisomorphism). For simplicity, we substitute  $\chi$  for  $\chi^{\text{na}}$  in the following. Indeed, when  $G = \text{id}$ , then the naïve Euler characteristic is just the Euler characteristic. So the overlapping notation should not cause any confusion. It is well-defined by the following observation.

Let  $[X/G](\mathbb{C})$  be the set of  $\mathbb{C}$ -points. A subset  $C \subseteq [X/G](\mathbb{C})$  is constructible if  $C = \bigcup_{i \in I} \mathcal{F}_i(\mathbb{C})$  where  $\{\mathcal{F}_i \mid i \in I\}$  is a finite set of algebraic  $\mathbb{C}$ -substacks of finite type of  $[X/G]$  (see [Jo1, Definition 4.1]).

**Lemma 3.8.** *Let  $X$  be a  $G$ -variety. If  $[X/G](\mathbb{C})$  is the disjoint union of finitely many constructible subsets  $Z_1, \dots, Z_r$ , then  $\chi(X/G) = \sum_{i=1}^r \chi(Z_i)$ .*

*Proof.* By the above construction, there is a pseudoisomorphism  $\Phi$  between  $[X/G](\mathbb{C})$  and  $Y := \coprod_{i=1}^l Y_i$ . For  $i = 1, \dots, r$ ,  $\Phi(Z_i)$  is a constructible subset of  $Y$  and  $Y = \bigsqcup_{i=1}^r \Phi(Z_i)$ . Then by the definition of the naïve Euler characteristic and Proposition 3.4, we deduce  $\chi(X/G) = \sum_{i=1}^r \chi(Z_i)$ .  $\square$

Let  $X$  and  $Y$  be two complex algebraic varieties under the actions of the algebraic groups  $G$  and  $H$ , respectively. Let  $\phi : [X/G] \rightarrow [Y/H]$  be a 1-morphism. Then it induces a natural pseudomorphism  $\phi_* : [X/G](\mathbb{C}) \rightarrow [Y/H](\mathbb{C})$  [Jo1, Proposition 4.6]. In the same way as the proof of Lemma 3.8, we obtain the following result.

**Lemma 3.9.** *If  $\phi_*$  is surjective and all fibres of  $\phi_*$  have the same naïve Euler characteristic  $\chi$ , then we have*

$$\chi(X/G) = \chi(Y/H) \cdot \chi.$$

We introduce the following notation. Let  $X$  be a variety under the action of an algebraic group  $G$ . Let  $f$  be a  $G$ -invariant constructible function over  $X$ . We define

$$\int_{x \in [X/G](\mathbb{C})} f(x) := \sum_{c \in \mathbb{C}} \chi(f^{-1}(c)/G)c.$$

**3.3.** Let  $\mathbf{d}_1, \mathbf{d}_2$  be two dimension vector in  $K_0$ . For any two subsets  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of  $\mathcal{P}^b(A, \mathbf{d}_1)$  and  $\mathcal{P}^b(A, \mathbf{d}_2)$  respectively, we define the subset  $\mathcal{O}_1 * \mathcal{O}_2$  of  $\mathcal{P}^b(A, \mathbf{d}_1 + \mathbf{d}_2)$  to be the set of  $z \in \mathcal{P}^b(A, \mathbf{d}_1 + \mathbf{d}_2)$  such that there exists a triangle

$$P(y) \longrightarrow P(z) \longrightarrow P(x) \longrightarrow P(x)[1]$$

in  $\mathcal{D}^b(A)$  where  $x \in \mathcal{O}_1$  and  $y \in \mathcal{O}_2$ . By the octahedral axiom, we have  $(\mathcal{O}_1 * \mathcal{O}_2) * \mathcal{O}_3 = \mathcal{O}_1 * (\mathcal{O}_2 * \mathcal{O}_3)$ . Inductively, we can define  $\mathcal{O}_1 * \mathcal{O}_2 * \dots * \mathcal{O}_s$  for all  $s > 1$ .



A subset  $\mathcal{O}$  of  $\mathcal{P}^b(A, \mathbf{d})$  is called a support-bounded constructible subset if there exists a constructible subset  $\mathcal{O}_{\underline{\mathbf{e}}}$  of  $\mathcal{P}^b(A, \underline{\mathbf{e}})$  for some  $\underline{\mathbf{d}}(\underline{\mathbf{e}}) \in \underline{\dim}^{-1}(\mathbf{d})$  such that  $\mathcal{O} = G_{\mathbf{d}}.t_{\underline{\mathbf{e}}}(\mathcal{O}_{\underline{\mathbf{e}}})$  where  $t_{\underline{\mathbf{e}}} : \mathcal{P}^b(A, \underline{\mathbf{e}}) \rightarrow \mathcal{P}^b(A, \mathbf{d})$  is canonical. Let  $\underline{\mathbf{d}}(\underline{\mathbf{e}}') \in \underline{\dim}^{-1}(\mathbf{d})$  and  $\underline{\mathbf{e}} \leq \underline{\mathbf{e}}'$ . We set  $\mathcal{O}_{\underline{\mathbf{e}}'} := G_{\underline{\mathbf{e}}'}.t_{\underline{\mathbf{e}}\underline{\mathbf{e}}'}(\mathcal{O}_{\underline{\mathbf{e}}})$ . Then we have

$$\mathcal{O} = \varinjlim_{\underline{\mathbf{d}}(\underline{\mathbf{e}}') \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{O}_{\underline{\mathbf{e}}'}.$$

The following proposition shows the property of support-bounded is invariant under derived equivalence.

**Proposition 3.10.** If two algebras  $A$  and  $B$  are derived equivalent, then this equivalent functor  $F$  induces the map  $f$  as in Theorem B sending constructible sets of support bounded in  $\mathcal{P}^b(A)$  to constructible sets of support bounded in  $\mathcal{P}^b(B)$ .

*Proof.* By Theorem B, the equivalent functor  $F : D^b(A) \rightarrow D^b(B)$  induces the following commutative diagram:

$$(3.1) \quad \begin{array}{ccc} \mathcal{P}^b(A, \mathbf{d}) & \xrightarrow{f} & \mathcal{P}^b(B, \mathbf{d}') \\ \pi_A \downarrow & & \downarrow \pi_B \\ \mathcal{Q}\mathcal{P}^b(A, \mathbf{d}) & \xrightarrow{\bar{f}} & \mathcal{Q}\mathcal{P}^b(B, \mathbf{d}') \end{array}$$

where  $f$  is a morphism of varieties and  $\bar{f}$  is a homeomorphism of quotient spaces. Let  $\mathcal{O}$  be a support-bounded constructible subset of  $\mathcal{P}^b(A, \mathbf{d})$ . Then  $\mathcal{O} = G_{\mathbf{d}}.\mathcal{O}_{\underline{\mathbf{e}}}$  for some projective dimension sequence  $\underline{\mathbf{e}}$  by definition. We have  $f(\mathcal{O}_{\underline{\mathbf{e}}})$  is a constructible subset of  $\mathcal{P}^b(B, \underline{\mathbf{e}}')$  for some projective dimension sequence  $\underline{\mathbf{e}}'$ . Since  $F$  is a derived equivalence,  $f(\mathcal{O})$  is  $G_{\mathbf{d}'}$ -invariant. On the other hand, due to the commutativity of the diagram,

$$f(\mathcal{O}) = \pi_B^{-1}\bar{f}\pi_A(\mathcal{O}) = \pi_B^{-1}\bar{f}\pi_A(\mathcal{O}_{\underline{\mathbf{e}}}) = G_{\mathbf{d}'}.\mathcal{O}_{\underline{\mathbf{e}}'}.$$

□

**Definition 3.11.** Let  $\mathbf{d}$  be a dimension vector in  $K_0(\mathcal{K}_0(\mathcal{D}^b(A)))$  and  $\mathcal{O}$  be a support-bounded constructible subset of  $\mathcal{P}^b(A, \mathbf{d})$ . Then we denote by  $\hat{1}_{\mathcal{O}}$  the  $\mathbb{C}$ -value function over  $\mathcal{P}^b(A, \mathbf{d})$  given by taking values 1 on each point in  $\mathcal{O}$  and 0 otherwise. A function  $\hat{f}$  on  $\mathcal{P}^b(A, \mathbf{d})$  is called a  $G_{\mathbf{d}}$ -invariant constructible function if  $\hat{f}$  can be written as a sum of finite terms  $\sum_i m_i \hat{1}_{\mathcal{O}_i}$  where  $m_i \in \mathbb{C}$  and  $\mathcal{O}_i$  is a support-bounded constructible subset of  $\mathcal{P}^b(A, \mathbf{d})$ . The set of  $G_{\mathbf{d}}$ -invariant constructible functions over  $\mathcal{P}^b(A, \mathbf{d})$  is denoted by  $M(\mathcal{P}^b(A, \mathbf{d}))$ .

Let  $\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})$  and  $M(\mathcal{P}^b(A, \underline{\mathbf{e}}))$  be the  $\mathbb{C}$ -space of  $G_{\underline{\mathbf{e}}}$ -invariant constructible functions over  $\mathcal{P}^b(A, \underline{\mathbf{e}})$ . For any  $\underline{\mathbf{e}}' \in \underline{\dim}^{-1}(\mathbf{d})$  and  $\underline{\mathbf{e}}' \geq \underline{\mathbf{e}}$ , there is a natural linear map  $r_{\underline{\mathbf{e}}'\underline{\mathbf{e}}} : M(\mathcal{P}^b(A, \underline{\mathbf{e}}')) \rightarrow M(\mathcal{P}^b(A, \underline{\mathbf{e}}))$  sending a function  $\hat{f}$  to  $t_{\underline{\mathbf{e}}\underline{\mathbf{e}}'} \circ \hat{f}$ . Then we obtain an inverse system.

**Proposition 3.12.** With the notations in Definition 3.11, we have

$$M(\mathcal{P}^b(A, \mathbf{d})) = \varprojlim_{\underline{\mathbf{d}}(\underline{\mathbf{e}}) \in \underline{\dim}^{-1}(\mathbf{d})} M(\mathcal{P}^b(A, \underline{\mathbf{e}})).$$

**3.4. Convolution.** Let  $\mathbf{d}', \mathbf{d}''$  be two dimension vectors in  $K_0$  and  $\mathbf{e}', \mathbf{e}''$  be two projective dimension vectors satisfying  $\underline{\mathbf{d}}(\mathbf{e}') \in \underline{\dim}^{-1}(\mathbf{d}')$ ,  $\underline{\mathbf{d}}(\mathbf{e}'') \in \underline{\dim}^{-1}(\mathbf{d}'')$ . The following two lemmas is easy corollaries of 4.1.13 in [GM].

**Lemma 3.13.** *If there is a distinguished triangle:  $X^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} Y^\bullet \xrightarrow{h} X^\bullet[1]$  for  $X^\bullet \in \mathcal{P}^b(A, \mathbf{e}'')$ ,  $Y^\bullet \in \mathcal{P}^b(A, \mathbf{e}')$ , and  $L^\bullet \in \mathcal{P}^b(A, \mathbf{e}' + \mathbf{e}'')$ , then it induces an exact sequence of chain complex:*

$$0 \longrightarrow X^\bullet \longrightarrow L^\bullet \longrightarrow Y^\bullet \longrightarrow 0$$

**Lemma 3.14.** *Let  $X^\bullet \in \mathcal{P}^b(A, \mathbf{e}'')$ ,  $Y^\bullet \in \mathcal{P}^b(A, \mathbf{e}')$  and the exact sequence:*

$$0 \longrightarrow X_i \xrightarrow{l_i} L_i \xrightarrow{\pi_i} Y_i \longrightarrow 0$$

*hold for all  $i$ -th terms of  $X^\bullet$ ,  $Y^\bullet$  and  $L^\bullet$  respectively, where  $l_i$  are the canonical injections and  $\pi_i$  the canonical projections. Define the set  $\{\partial^L\}$  to be the set of sequences  $(\partial_i^L)_i$  such that  $L^\bullet = (X_i \oplus Y_i, \partial_i^L)_i$  becomes a complex and induces the exact sequence of complexes*

$$0 \longrightarrow X^\bullet \longrightarrow L^\bullet \longrightarrow Y^\bullet \longrightarrow 0.$$

*Then we have the canonical isomorphism  $\{\partial^L\} \cong \text{Hom}_{\mathcal{K}^b(A)}(Y^\bullet[-1], X^\bullet)$ .*

Let  $\mathcal{O}_1 \subset \mathcal{P}^b(A, \mathbf{e}'')$  and  $\mathcal{O}_2 \subset \mathcal{P}^b(A, \mathbf{e}')$  be  $G_{\mathbf{e}''}$ - and  $G_{\mathbf{e}'}$ -invariant constructible set, respectively. Put

$$L(\mathcal{O}_1, \mathcal{O}_2) = \{Cone(h) \mid h \in \text{Hom}_{\mathcal{K}^b(\mathcal{P}(A))}(\mathcal{O}_1[-1], \mathcal{O}_2)\}.$$

For  $L \in \mathcal{P}^b(A, \mathbf{e}' + \mathbf{e}'')$ , we set

$$W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2; L) = \{(f, g, h) \mid Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1] \text{ is a distinguished triangle,}$$

$$\text{with } X \in \mathcal{O}_1, Y \in \mathcal{O}_2\}$$

$$W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2) = \{(L, (f, g, h)) \mid L \in \mathcal{P}^b(A, \mathbf{e}' + \mathbf{e}''), (f, g, h) \in W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)\}$$

and

$$\text{Hom}_{\mathcal{K}^b(\mathcal{P}(A))}(\mathcal{O}_1[-1], \mathcal{O}_2) = \{(Y, X, h) \mid X \in \mathcal{O}_1, Y \in \mathcal{O}_2, h \in \text{Hom}_{\mathcal{K}^b(\mathcal{P}(A))}(X[-1], Y)\}.$$

We introduce the action of  $G_{\mathbf{e}''} \times G_{\mathbf{e}'}$  on  $W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2)$  and  $\text{Hom}_{\mathcal{K}^b(\mathcal{P}(A))}(\mathcal{O}_1[-1], \mathcal{O}_2)$  as follows:

For  $(a, c) \in G_{\mathbf{e}''} \times G_{\mathbf{e}'}$ ,

$$(a, c) \circ (L, (f, g, h)) = (L, (fc^{-1}, ag, (c[1])ha)),$$

$$(a, c) \circ (h) = ch(a[-1])^{-1}.$$

The action of  $G_{\mathbf{e}''} \times G_{\mathbf{e}'}$  on  $W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2)$  naturally induces the action on  $W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)$ . We consider the naïve Euler characteristic of quotient space  $W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\mathbf{e}''} \times G_{\mathbf{e}'}$  and have the following result.

**Proposition 3.15.** Let  $\mathcal{O}_1, \mathcal{O}_2$  be as above. Then the set

$$\{\chi(W_{\mathbf{e}'\mathbf{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\mathbf{e}''} \times G_{\mathbf{e}'} \mid L \in \mathcal{P}^b(A, \mathbf{e}' + \mathbf{e}'')\}$$

is a finite set.

*Proof.* Consider the quotient stack  $[W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2)/G_{\underline{e}''} \times G_{\underline{e}'}]$  of  $W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2)$  under the action of  $G_{\underline{e}''} \times G_{\underline{e}'}$ . As in Section 3.2, there exists a pseudoisomorphism  $\Phi$  between  $[W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2)/G_{\underline{e}''} \times G_{\underline{e}'}](\mathbb{C})$  and some variety  $Y$ . The natural projection  $\pi : W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2) \rightarrow \mathcal{P}^b(A, \underline{e}' + \underline{e}'')$  and  $\Phi^{-1}$  combines to give a pseudomorphism  $\Psi : Y \rightarrow \mathcal{P}^b(A, \underline{e}' + \underline{e}'')$ . It is a morphism of varieties [Jo1, Section 3.4]. By Theorem 3.5,  $\pi_*(1_Y)$  is constructible. By the definition of pushforward functor, it means that the set

$$\{\chi(\Psi^{-1}(L)) \mid L \in \mathcal{P}^b(A, \underline{e}' + \underline{e}'')\}$$

is a finite set. Therefore the set  $\{\chi(W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\underline{e}''} \times G_{\underline{e}'} \mid L \in \mathcal{P}^b(A, \underline{e}' + \underline{e}'')\}$  is finite.  $\square$

**Definition 3.16.** Let  $\mathcal{O}_1 \subset \mathcal{P}^b(A, \underline{e}'')$  and  $\mathcal{O}_2 \subset \mathcal{P}^b(A, \underline{e}')$  be  $G_{\underline{e}''}$ - and  $G_{\underline{e}'}$ -invariant constructible sets, respectively. The convolution multiplication  $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2} \in M(\mathcal{P}^b(A, \underline{e}' + \underline{e}''))$  is defined as follows:

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}(L) = \chi(W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\underline{e}''} \times G_{\underline{e}'})$$

for  $L \in \mathcal{P}^b(A, \underline{e}' + \underline{e}'')$ . We set  $V(\mathcal{O}_1, \mathcal{O}_2) := [W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2)/G_{\underline{e}''} \times G_{\underline{e}'}]$  and  $V_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L) := [W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\underline{e}''} \times G_{\underline{e}'}]$ . Write  $F_{\mathcal{O}_1, \mathcal{O}_2}^L = \chi(V_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L))$ .

Obviously  $1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}$  is again  $G_{\underline{e}' + \underline{e}''}$ -invariant. In this way, the proof of Proposition 3.15 implies

**Corollary 3.17.** If  $f \in M(\mathcal{P}^b(A, \underline{e}''))$  and  $g \in M(\mathcal{P}^b(A, \underline{e}'))$ , then the convolution  $f * g \in M(\mathcal{P}^b(A, \underline{e}' + \underline{e}''))$  is well-defined.

The above discussion can be extended to  $\mathcal{P}^b(A, \mathbf{d})$ . Assume that  $\underline{e}' \in \underline{\dim}^{-1}(\mathbf{d}')$  and  $\underline{e}'' \in \underline{\dim}^{-1}(\mathbf{d}'')$ . Put  $\mathcal{O} = G_{\mathbf{d}''}.t_{\underline{e}''}(\mathcal{O}_1)$  and  $\mathcal{O}' = G_{\mathbf{d}'} . t_{\underline{e}'}(\mathcal{O}_2)$ . Let  $\underline{e}'_0 \geq \underline{e}'$  and  $\underline{e}''_0 \geq \underline{e}''$  be in  $\underline{\dim}^{-1}(\mathbf{d}')$  and  $\underline{\dim}^{-1}(\mathbf{d}'')$ , respectively. Then  $t_{\underline{e}'\underline{e}''_0}$  and  $t_{\underline{e}''\underline{e}''_0}$  naturally induces a map  $t_{\underline{e}'\underline{e}''_0, \underline{e}''\underline{e}''_0} : W_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2) \rightarrow W_{\underline{e}'_0\underline{e}''_0}(\mathcal{O}_{1\underline{e}'_0}, \mathcal{O}_{2\underline{e}''_0})$  where  $\mathcal{O}_{1\underline{e}'_0} = G_{\underline{e}'_0}.t_{\underline{e}'\underline{e}'_0}(\mathcal{O}_1)$  and  $\mathcal{O}_{2\underline{e}''_0} = G_{\underline{e}''_0}.t_{\underline{e}''\underline{e}''_0}(\mathcal{O}_2)$ . Then we obtain a direct system and set

$$W(\mathcal{O}, \mathcal{O}') = \varinjlim W_{\underline{e}'_0\underline{e}''_0}(\mathcal{O}_{1\underline{e}'_0}, \mathcal{O}_{2\underline{e}''_0}).$$

For any  $L \in \mathcal{P}^b(A, \mathbf{d}' + \mathbf{d}'')$ , we set  $L_{\underline{e}} := t_{\underline{e}}^{-1}(L) \in \mathcal{P}^b(A, \underline{e})$  (perhaps it does not exist!). Then we define

$$W(\mathcal{O}, \mathcal{O}'; L) = \varinjlim W_{\underline{e}'_0\underline{e}''_0}(\mathcal{O}_{1\underline{e}'_0}, \mathcal{O}_{2\underline{e}''_0}; L_{\underline{e}'_0 + \underline{e}''_0}).$$

There are natural actions of  $G_{\mathbf{d}''} \times G_{\mathbf{d}'}$  on  $W(\mathcal{O}, \mathcal{O}'; L)$  and  $W(\mathcal{O}, \mathcal{O}')$ . The orbit spaces are denoted by  $V(\mathcal{O}, \mathcal{O}'; L)$  and  $V(\mathcal{O}, \mathcal{O}')$ , respectively. We note that  $V(\mathcal{O}, \mathcal{O}'; L) = V_{\underline{e}'\underline{e}''}(\mathcal{O}_1, \mathcal{O}_2; L_{\underline{e}' + \underline{e}''})$ .

**Definition 3.18.** Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two support-bounded constructible subsets of  $\mathcal{P}^b(A, \mathbf{d}'')$  and  $\mathcal{P}^b(A, \mathbf{d}')$ , respectively. Then we define

$$\hat{1}_{\mathcal{O}_1} * \hat{1}_{\mathcal{O}_2}(L) = \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$$

for  $L \in \mathcal{P}^b(A, \mathbf{d}' + \mathbf{d}'')$  and set  $F_{\mathcal{O}_1, \mathcal{O}_2}^L = \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$ .

As Corollary 3.17, we have

**Proposition 3.19.** If  $\hat{f} \in M(\mathcal{P}^b(A, \mathbf{d}''))$  and  $\hat{g} \in M(\mathcal{P}^b(A, \mathbf{d}'))$ , then  $\hat{f} * \hat{g} \in M(\mathcal{P}^b(A, \mathbf{d}' + \mathbf{d}''))$  is well-defined.

Let  $\hat{f} \in M(\mathcal{P}^b(A, \mathbf{d}''))$  and  $\hat{g} \in M(\mathcal{P}^b(A, \mathbf{d}'))$ . Then

$$(\hat{f} * \hat{g})(x) = \int_{\bar{U}} \hat{f}(x') \hat{g}(x'') := \sum_{c, d \in \mathbb{C}} \chi(V(f^{-1}(c), g^{-1}(d); x)cd.$$

where  $\bar{U} = V(\text{supp}(f), \text{supp}(g); x)$ .

*Remark 3.20.* The definition 3.18 does not supply an associative multiplication in general. In [T], the author define an associative multiplication for the derived category  $\mathcal{D}^b(A)$  over a finite field. However, it is not known how to make an analogy of this associative multiplication over  $\mathbb{C}$ .

#### 4. THE RELATIVE HOMOTOPY CATEGORY OF $m$ -CYCLE COMPLEXES

Let  $A$  be finite dimensional and finite global dimensional associative algebra over  $\mathbb{C}$  and  $m$  be a positive even number. We will recall some results in [PX2] for the relative homotopy category of  $m$ -cycle complexes over  $A$  and define their geometrization.

4.1. A  $m$ -cycle complex over  $A$  is by definition a complex  $X^\bullet = (X_i, \partial_i)$  satisfying  $X_i = X_j$  and  $\partial_i = \partial_j$  for all  $i, j \in \mathbb{Z}$  with  $i \equiv j \pmod{m}$ . If  $X^\bullet$  and  $Y^\bullet$  are two  $m$ -cycle complexes, a *morphism*  $f^\bullet : X^\bullet \rightarrow Y^\bullet$  is a morphism of complexes such that  $f_i = f_j$  for all  $i, j \in \mathbb{Z}$  with  $i \equiv j \pmod{m}$ . Hence, all  $m$ -cycle complexes constitute an abelian subcategory of  $\mathcal{C}(A)$ , denoted by  $\mathcal{C}_m(A)$ . We also denote the subcategory of  $\mathcal{C}_m(A)$  consisting of  $m$ -cycle complex whose term is projective  $A$ -module by  $\mathcal{P}_m(A)$ .

Let  $f^\bullet, g^\bullet : X^\bullet \rightarrow Y^\bullet$  be two morphisms of  $m$ -cycle complexes. A *relative homotopy*  $s^\bullet$  from  $f^\bullet$  to  $g^\bullet$  is a homotopy map of complex such that  $s_i = s_j$  for all  $i, j \in \mathbb{Z}$  with  $i \equiv j \pmod{m}$ . Under this condition,  $f^\bullet$  and  $g^\bullet$  are said to be relatively homotopic. Hence, we can form an additive and homotopy category  $\mathcal{K}_m(A)$ . We use this notation though it usually is not a subcategory of  $\mathcal{K}(A)$ . We also denote  $\mathcal{P}(A) \cap \mathcal{C}_m(A)$  and  $\mathcal{K}(\mathcal{P}(A)) \cap \mathcal{K}_m(A)$  by  $\mathcal{P}_m(A)$  and  $\mathcal{K}_m(\mathcal{P}(A))$ .

We define a functor  $CF : \mathcal{C}^b(A) \rightarrow \mathcal{C}_m(A)$  as follows. For  $X^\bullet = (X_i, \partial_i^X) \in \mathcal{C}^b(A)$ , set  $FX^\bullet = ((FX^\bullet)_i, \partial_i^{FX})$  where  $(FX^\bullet)_i = \bigoplus X_{i+tm}$  and  $\partial_i^{FX} = (\partial_{i,s,t}^X)_{s,t \in \mathbb{Z}}$  such that  $\partial_{i,s,t}^X : X_{i+sm} \rightarrow X_{i+1+tm}$  with  $\partial_{i,s,t}^X = 0$  for  $s \neq t$  and  $\partial_{i,s,t}^X = \partial_i^X$ .  $CF$  induces a functor  $F : \mathcal{K}^b(\mathcal{P}(A)) \rightarrow \mathcal{K}_m(\mathcal{P}(A))$ .

**Theorem 4.1.**  $\mathcal{K}_m(\mathcal{P}(A))$  is a triangulated category with the shift functor defined for complex category and the functor  $F : \mathcal{K}^b(\mathcal{P}(A)) \rightarrow \mathcal{K}_m(\mathcal{P}(A))$  is exact.

For the natural functors:

$$\mathcal{K}^b(\mathcal{P}(A)) \rightarrow \mathcal{K}^b(A) \rightarrow \mathcal{D}^b(A)$$

the functors  $F$  induce the following commutative diagram

$$\begin{array}{ccccc} \mathcal{K}^b(\mathcal{P}(A)) & \longrightarrow & \mathcal{K}^b(A) & \longrightarrow & \mathcal{D}^b(A) \\ \downarrow F & & \downarrow F & & \downarrow F \\ \mathcal{K}_m(\mathcal{P}(A)) & \longrightarrow & \mathcal{K}_m(A) & \longrightarrow & \mathcal{D}_m(A) \end{array}$$

where  $\mathcal{D}_m(A)$  is the  $m$ -periodic derived category of mod  $A$ . It is known that  $\mathcal{K}_m(\mathcal{P}(A))$  is a triangulated full subcategory of  $\mathcal{K}_m(A)$ . Therefore we can regard  $\mathcal{K}_m(\mathcal{P}(A))$

as a triangulated full subcategory of  $\mathcal{D}_m(A)$ . We consider the triangulated full subcategory  $\mathcal{R}_m(A)$  of  $\mathcal{D}_m(A)$  (also of  $\mathcal{K}_m(\mathcal{P}(A))$ ) generated by the full subcategory  $F(\mathcal{K}^b(\mathcal{P}(A)))$  (also by  $\mathcal{K}^b(A)$ ). In general, the functor  $F$  is not dense. If  $A$  is hereditary, then  $\mathcal{R}_m(A) = \mathcal{K}_m(A)$ . When  $m = 2$  we call  $\mathcal{R}_2(A)$  the root category of mod  $A$ . By the way, the Galois group associated with  $F$  is the cyclic group generated by  $T^m$ .

4.2. A complex  $C^\bullet$  of  $A$ -modules is called period-2 (or 2-periodic) complex if it satisfies  $C^\bullet[2] = C^\bullet$ . We can simply write the sequence of dimension vector of  $C^\bullet$  as  $\underline{d}(C^\bullet) = (\underline{d}(C^0), \underline{d}(C^1))$ . For a sequence of dimension vectors  $\underline{d} = (\underline{d}_0, \underline{d}_1)$ , we define  $\mathcal{C}_2(A, \underline{d})$  to be the subset of

$$\mathbb{E}_{\underline{d}_0}(Q, R) \times \mathbb{E}_{\underline{d}_1}(Q, R) \times \text{Hom}_{\mathbb{C}}(V^{\underline{d}_0}, V^{\underline{d}_1}) \times \text{Hom}_{\mathbb{C}}(V^{\underline{d}_1}, V^{\underline{d}_0})$$

which consists of elements  $x = (x^0, x^1, \partial_0, \partial_1)$ , where  $x^i \in \mathbb{E}_{\underline{d}_i}(Q, R)$  and  $M(x^i)$  are the corresponding  $A$ -modules on the space  $V^{\underline{d}_i}$  for  $i = 0, 1$  respectively; and  $\partial_0 \in \text{Hom}_{\mathbb{C}}(V^{\underline{d}_0}, V^{\underline{d}_1})$  is a  $A$ -module homomorphism from  $M(x^0)$  to  $M(x^1)$ ,  $\partial_1 \in \text{Hom}_{\mathbb{C}}(V^{\underline{d}_1}, V^{\underline{d}_0})$  is a  $A$ -module homomorphism from  $M(x^1)$  to  $M(x^0)$  with the property  $\partial_1 \partial_0 = 0$  and  $\partial_0 \partial_1 = 0$ . As in [LP], a 2-periodic complex  $C^\bullet$  can be written as

$$C^0 \begin{array}{c} \xrightarrow{\partial_0} \\ \xleftarrow{\partial_1} \end{array} C^1$$

with  $\partial_0 \partial_1 = \partial_1 \partial_0 = 0$ .

The group  $G_{\underline{d}}(Q) = GL_{\underline{d}_0}(Q) \times GL_{\underline{d}_1}(Q)$  acts on  $\mathcal{C}_2(A, \underline{d})$  via the conjugation action

$$(g_i)_i(x_i, \partial_i)_i = ((x_i)^{g_i}, g_{i-1} \partial_i g_i^{-1})_i$$

where the action  $(x_i)^{g_i}$  was defined as in Section 1.1. For  $x \in \mathcal{C}_2(A, \underline{d})$ , we denote its corresponding complex by  $M^\bullet(x)$ . Therefore the orbits under the action correspond bijectively to the isomorphism classes of complexes of period-2  $A$ -modules.

Giving the complete set  $P_1, P_2, \dots, P_l$  of indecomposable projective  $A$ -modules as in Section 1. Let  $P^\bullet = (P^0, P^1, \partial_0, \partial_1)$  be a period-2 complex where  $P^0 \cong \bigoplus_{j=1}^l e_j^0 P_j$  and  $P^1 \cong \bigoplus_{j=1}^l e_j^1 P_j$ . Let  $\underline{e}(P^i) = (e_j^i)$  for  $i = 0, 1$ . Then  $\underline{e} = (\underline{e}(P^0), \underline{e}(P^1))$  is the projective dimension sequence of  $P^\bullet$ . Write  $\underline{d}(\underline{e}) = (\underline{\dim} P^0, \underline{\dim} P^1)$ . We define  $\mathcal{P}_2(A, \underline{e})$  to be the subset of  $\mathcal{C}_2(A, \underline{d}(\underline{e}))$  consisting of elements  $x = (x^0, x^1, \partial_0, \partial_1)$ , where  $(\mathbb{C}^{\underline{d}_i}, x^i) \cong P^i$  for  $i = 0, 1$ .

The algebraic group  $G_{\underline{d}(\underline{e})}(Q, R)$  acts on  $\mathcal{P}_2(A, \underline{e})$  by

$$(g_i)_i(\partial_i)_i = (g_{i-1} \partial_i g_i^{-1})_i.$$

For  $x \in \mathcal{P}_2(A, \underline{e})$ , we denote the corresponding complex in  $\mathcal{P}_2(A)$  by  $P^\bullet(x)$ .

4.3. A complex  $X^\bullet = (X_i, \partial_i)_i \in \mathcal{P}_2(A)$  is called contractible if the induced homological groups  $H_i(X^\bullet) = \ker \partial_{i+1} / \text{Im } \partial_i = 0$  for all  $i = 0, 1$ . It is easy to see that any contractible projective complex is isomorphic to a direct sum of shifted copies of complexes of the form

$$(4.1) \quad \dots \xrightarrow{f} P \xrightarrow{0} P \xrightarrow{f} P \xrightarrow{0} P \xrightarrow{f} \dots$$

with  $P$  a projective  $A$ -module and  $f$  an automorphism. We call an element  $x \in \mathcal{P}_2(A, \underline{e})$  contractible if the corresponding projective complex is contractible. As in Section 2.3, we can prove the 2-periodic versions for Lemma 2.2, Lemma 2.3 and Lemma 2.4. Any complex in  $\mathcal{P}_2(A)$  is isomorphic to the direct sum of minimal

period-2 projective complex and a contractible projective complex. Hence, we can define a direct system  $\{(\mathcal{P}_2(A, \underline{\mathbf{e}}), t_{\underline{\mathbf{e}}\underline{\mathbf{e}}'}) \mid \underline{\mathbf{e}}, \underline{\mathbf{e}}' \in \underline{\dim}^{-1}(\mathbf{d}), \underline{\mathbf{e}} \leq \underline{\mathbf{e}}'\}$  for any  $\mathbf{d} \in K_0$  where  $K_0$  is the Grothendieck group of the category  $\mathcal{K}_2(\mathcal{P}(A))$ . We note that all triangulated categories  $\mathcal{K}_2(A)$ ,  $\mathcal{K}_2(\mathcal{P}(A))$ ,  $\mathcal{D}_2(A)$ ,  $\mathcal{K}^b(\mathcal{P}(A))$  and  $\mathcal{D}^b(A)$  have the same Grothendieck groups. Then we define

$$\mathcal{P}_2(A, \mathbf{d}) := \varinjlim_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{P}_2(A, \underline{\mathbf{e}}) \quad \text{and} \quad G_{\mathbf{d}} = \varinjlim_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})} G_{\underline{\mathbf{d}}(\underline{\mathbf{e}})}.$$

We also have the quotient space

$$\mathcal{QP}_2(A, \mathbf{d}) = \mathcal{P}_2(A, \mathbf{d}) / \sim,$$

where  $x \sim y$  in  $\mathcal{P}_2(A, \mathbf{d})$  if and only if the corresponding projective complexes  $P^\bullet(x)$  and  $P^\bullet(y)$  are quasi-isomorphic, i.e., they are isomorphic in  $\mathcal{K}_2(\mathcal{P}(A))$ . We denote by  $\mathcal{QP}_2(A, \underline{\mathbf{e}})$  and  $\mathcal{Q}_1\mathcal{P}_2(A, \mathbf{d})$  the orbit spaces of  $\mathcal{P}_2(A, \underline{\mathbf{e}})$  and  $\mathcal{P}_2(A, \mathbf{d})$  under the actions of  $G_{\underline{\mathbf{d}}(\underline{\mathbf{e}})}$  and  $G_{\mathbf{d}}$ , respectively. As Proposition 2.1, we have the follow analogy.

**Proposition 4.2.** With the above notations, we have

$$\mathcal{QP}_2(A, \mathbf{d}) = \mathcal{Q}_1\mathcal{P}_2(A, \mathbf{d}) = \varinjlim_{\underline{\mathbf{d}}(\underline{\mathbf{e}}) \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{QP}_2(A, \underline{\mathbf{e}}).$$

4.4. We can define the 2-periodic versions of all notations in Section 3.4 by substituting  $\mathcal{P}_2(A, \underline{\mathbf{e}})$  and  $\mathcal{P}_2(A, \mathbf{d})$  for  $\mathcal{P}^b(A, \underline{\mathbf{e}})$  and  $\mathcal{P}^b(A, \mathbf{d})$ , respectively. For  $\mathbf{d} \in K_0$  and  $\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})$ , let  $\mathcal{O} \subseteq \mathcal{P}_2(A, \underline{\mathbf{e}})$  be a  $G_{\underline{\mathbf{e}}}$ -invariant constructible subset. Then  $G_{\mathbf{d}}\mathcal{O}$  be a support-bounded constructible subset. We denote by  $\hat{1}_{\mathcal{O}}$  the  $\mathbb{C}$ -value function over  $\mathcal{P}^b(A, \mathbf{d})$  given by taking values 1 on each point in  $\mathcal{O}$  and 0 otherwise. A function  $\hat{f}$  on  $\mathcal{P}_2(A, \mathbf{d})$  is called a  $G_{\mathbf{d}}$ -invariant constructible function if  $\hat{f}$  can be written as a sum of finite terms  $\sum_i m_i \hat{1}_{\mathcal{O}_i}$  where  $m_i \in \mathbb{C}$  and  $\mathcal{O}_i$  is a support-bounded constructible subset of  $\mathcal{P}_2(A, \mathbf{d})$ . We denote by  $M(\mathcal{P}_2(A, \mathbf{d}))$  the set of  $G_{\mathbf{d}}$ -invariant constructible functions.

**Definition 4.3.** Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two support-bounded constructible subsets of  $\mathcal{P}_2(A, \mathbf{d}'')$  and  $\mathcal{P}_2(A, \mathbf{d}')$ , respectively. Then we define

$$\hat{1}_{\mathcal{O}_1} * \hat{1}_{\mathcal{O}_2}(L) = \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$$

and set  $F_{\mathcal{O}_1, \mathcal{O}_2}^L = \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$  for  $L \in \mathcal{P}_2(A, \mathbf{d}' + \mathbf{d}'')$  where  $V(\mathcal{O}_1, \mathcal{O}_2; L)$  is a 2-periodic analogue of  $V(\mathcal{O}_1, \mathcal{O}_2; L)$  defined in Section 3.4.

As Corollary 3.17, we have

**Proposition 4.4.** If  $\hat{f} \in M(\mathcal{P}_2(A, \mathbf{d}''))$  and  $\hat{g} \in M(\mathcal{P}_2(A, \mathbf{d}'))$ , then  $\hat{f} * \hat{g} \in M(\mathcal{P}_2(A, \mathbf{d}' + \mathbf{d}''))$  is well-defined.

Let  $\hat{f} \in M(\mathcal{P}_2(A, \mathbf{d}''))$  and  $\hat{g} \in M(\mathcal{P}_2(A, \mathbf{d}'))$ . Then

$$(\hat{f} * \hat{g})(x) = \int_{\bar{U}} \hat{f}(x') \hat{g}(x'') := \sum_{c, d \in \mathbb{C}} \chi(V(f^{-1}(c), g^{-1}(d); x)cd.$$

where  $\bar{U} = V(\text{supp}(f), \text{supp}(g); x)$ .

## 5. REALIZATION OF LIE ALGEBRAS

Let  $\mathbf{d}_1, \mathbf{d}_2$  and  $\mathbf{d}$  be in  $K_0$  and  $\mathcal{O}_1 \in \mathcal{P}_2(A, \mathbf{d}_1)$ ,  $\mathcal{O}_2 \in \mathcal{P}_2(A, \mathbf{d}_2)$  and  $\mathcal{O} \in \mathcal{P}_2(A, \mathbf{d})$  be support-bounded constructible sets. Then  $\mathcal{O}_1 = G_{\mathbf{d}_1} \cdot \mathcal{O}_{\mathbf{e}''}$ ,  $\mathcal{O}_2 = G_{\mathbf{d}_2} \cdot \mathcal{O}_{\mathbf{e}'}$  and  $\mathcal{O} = G_{\mathbf{d}} \cdot \mathcal{O}_{\mathbf{e}}$  for projective dimension sequences  $\mathbf{e}''$ ,  $\mathbf{e}'$  and  $\mathbf{e}$ .

A constructible set  $\mathcal{O}$  is called indecomposable if all points in  $\mathcal{O}$  correspond to indecomposable objects in  $\mathcal{K}_2(\mathcal{P}(A))$ .

5.1. Let  $\mathcal{L} = \mathcal{L}(\mathcal{O}_1, \mathcal{O}_2)$

$= \{L \in \mathcal{K}_2(\mathcal{P}(A)) \mid \text{there exists triangle } Y \rightarrow L \rightarrow X \rightarrow Y[1] \text{ with } X \in \mathcal{O}_1, Y \in \mathcal{O}_2\}$

Then  $\mathcal{L}$  is a support-bounded constructible set in  $\mathcal{P}_2(A, \mathbf{d})$  and  $\mathcal{L} = G_{\mathbf{d}} \cdot \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}$  for some constructible subset  $\mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}$  in  $\mathcal{P}_2(A, \mathbf{e}'' + \mathbf{e}')$ . We will consider the following quotient stacks.

(a) Let  $W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}, \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}) := \bigcup_{L \in \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}} W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}, L)$ . The action of  $G_{\mathbf{e}''} \times G_{\mathbf{e}'}$  on  $W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}, \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'})$  is defined as in Section 3.4. For  $(a, c) \in G_{\mathbf{e}''} \times G_{\mathbf{e}'}$ ,  $(L, (f, g, h)) \in W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}, \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'})$ , define

$$(a, c) \circ (L, (f, g, h)) = (L, (fc^{-1}, ag, (c[1])ha^{-1})).$$

The quotient stack is  $V(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}; \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}) := [W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}; \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'})/G_{\mathbf{e}''} \times G_{\mathbf{e}'}]$ . We denote by  $(L, (f, g, h))^\wedge$  the geometric point in  $V(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}; \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'})$  corresponding to  $(L, (f, g, h))$ . Moreover, up to 1-isomorphism, the quotient stack is independent of the choice  $\mathbf{e}''$  and  $\mathbf{e}'$  as in Section 3.4. Hence, we also denote it by  $V(\mathcal{O}_1, \mathcal{O}_2; \mathcal{L}) := V(\mathcal{O}_{1\mathbf{e}'}, \mathcal{O}_{2\mathbf{e}'}; \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'})$  and set  $F_{\mathcal{O}_1 \mathcal{O}_2}^{\mathcal{L}} = \chi(V(\mathcal{O}_1, \mathcal{O}_2; \mathcal{L}))$ .

(b) Let  $M \in \mathcal{P}_2(\mathbf{e}'' + \mathbf{e}' + \mathbf{e})$  and

$$W_{(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}), \mathcal{O}_{\mathbf{e}}}^{\mathcal{L}_{\mathbf{e}'' + \mathbf{e}'} M} := \bigcup_{L \in \mathcal{L}_{\mathbf{e}'' + \mathbf{e}'}} W(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}; L) \times W(L, \mathcal{O}_{\mathbf{e}}; M).$$

Consider the action of  $G_{\mathbf{e}''} \times G_{\mathbf{e}' + \mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}}$ . For  $(a, b, c, d) \in G_{\mathbf{e}''} \times G_{\mathbf{e}' + \mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}}$  and  $(L, (f, g, h), (l, m, n)) \in W_{(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}), \mathcal{O}_{\mathbf{e}}}^{\mathcal{L}_{\mathbf{e}'' + \mathbf{e}'} M}$ , define

$$(a, b, c, d) \circ (L, (f, g, h), (l, m, n)) = (L', (bfc^{-1}, agb^{-1}, c[1]ha^{-1}), (ld^{-1}, bm, d[1]nb^{-1})).$$

The quotient stack is denoted by  $(W_{(\mathcal{O}_{\mathbf{e}''}, \mathcal{O}_{\mathbf{e}'}), \mathcal{O}_{\mathbf{e}}}^{\mathcal{L}_{\mathbf{e}'' + \mathbf{e}'} M})^\wedge$ . We denote by

$$(L, (f, g, h), (l, m, n))^\wedge = \{(b(L), (bfc^{-1}, agb^{-1}, c[1]ha^{-1}), (ld^{-1}, bm, d[1]nb^{-1})) \mid (a, b, c, d) \in G_{\mathbf{e}''} \times G_{\mathbf{e}' + \mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}}\}$$

the geometric point corresponding to  $(L, (f, g, h), (l, m, n))$ . The quotient stack is also independent of the choices of  $\mathbf{e}''$ ,  $\mathbf{e}'$  and  $\mathbf{e}$ . Depending on the discussion in Section 3.4, we can denote it by  $(W_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M})^\wedge$  and set  $\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M} = \chi((W_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M})^\wedge)$ .

Dually, let

$$W_{\mathcal{O}_{\mathbf{e}''}(\mathcal{O}_{\mathbf{e}'}), \mathcal{O}_{\mathbf{e}}}^{M \mathcal{L}'_{\mathbf{e} + \mathbf{e}'}} := \bigcup_{L' \in \mathcal{L}'_{\mathbf{e} + \mathbf{e}'}} W(\mathcal{O}_{\mathbf{e}}, \mathcal{O}_{\mathbf{e}'}; L') \times W(\mathcal{O}_{\mathbf{e}'}, L'; M).$$

There is also an action of  $G_{\mathbf{e}''} \times G_{\mathbf{e}' + \mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}}$  as above. For  $(a, b', c, d) \in G_{\mathbf{e}''} \times G_{\mathbf{e}' + \mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}}$  and  $(L', (f', g', h'), (l', m', n')) \in W_{\mathcal{O}_{\mathbf{e}''}(\mathcal{O}_{\mathbf{e}'}), \mathcal{O}_{\mathbf{e}}}^{M \mathcal{L}'_{\mathbf{e} + \mathbf{e}'}}$ , define

$$(a, b', c, d) \circ (L', (f', g', h'), (l', m', n')) = (L'', (b'l'd^{-1}, cm'b'^{-1}, (d[1])n'c^{-1}), (f'b'^{-1}, ag', (b'[1])h'a^{-1})).$$

The quotient stack is denoted by  $(W_{\mathcal{O}_{\underline{e}''}(\mathcal{O}_{\underline{e}'}\mathcal{O}_{\underline{e}})}^{M\mathcal{L}'^{\underline{e}+\underline{e}'}})^{\wedge}$ . We denote by

$$(L', (f', g', h'), (l', m', n'))^{\wedge}$$

the geometric point corresponding to  $(L', (f', g', h'), (l', m', n'))$ . The quotient stack is also independent of the choices of  $\underline{e}''$ ,  $\underline{e}'$  and  $\underline{e}$ . Depending on the discussion in Section 3.4, we can denote it by  $(W_{\mathcal{O}_{\underline{e}''}(\mathcal{O}_{\underline{e}'}\mathcal{O}_{\underline{e}})}^{M\mathcal{L}'^{\underline{e}+\underline{e}'}})^{\wedge}$  and set  $\chi_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}'} = \chi((W_{\mathcal{O}_{\underline{e}''}(\mathcal{O}_{\underline{e}'}\mathcal{O}_{\underline{e}})}^{M\mathcal{L}'^{\underline{e}+\underline{e}'}})^{\wedge})$ .

(c) The group  $G_{\underline{e}}$  acts on  $\tilde{W}(\mathcal{L}_{\underline{e}''+\underline{e}'}, \mathcal{O}_{\underline{e}}; M)$  as follows. For any  $d \in G_{\underline{e}}$ ,  $(l, m, n) \in W(\mathcal{L}_{\underline{e}''+\underline{e}'}, \mathcal{O}_{\underline{e}}; M)$ ,

$$d \circ (l, m, n) = (ld^{-1}, m, d[1]n)$$

The quotient stack is denoted by  $W(\mathcal{L}_{\underline{e}''+\underline{e}'}, \mathcal{O}_{\underline{e}}; M)^*$ . It is independent of the choice of  $\underline{e}$ . Then we can also write  $W(\mathcal{L}_{\underline{e}''+\underline{e}'}, \mathcal{O}; M)^*$ . We denote by  $(l, m, n)^* = \{ld^{-1}, m, d[1]n \mid d \in G_{\underline{e}}\}$  the geometric point corresponding to  $(l, m, n)$ .

5.2. Let  $K_0$  be the Grothendieck group of  $\mathcal{K}_2(\mathcal{P}(A))$ . Write  $\mathfrak{h} \simeq K_0 \otimes_{\mathbb{Z}} \mathbb{C}$ , which is spanned by  $\{h_{\mathbf{d}} \mid \mathbf{d} \in K_0\}$  subject to the relation:  $h_{\mathbf{d}} = h_{\mathbf{d}_1} + h_{\mathbf{d}_2}$  if  $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$  in  $K_0$ . For  $\mathcal{O} \subset \mathcal{P}_2(A, \mathbf{d})$ , we write  $h_{\mathcal{O}} := h_{\mathbf{d}}$ . The symmetric Euler bilinear form on  $\mathfrak{h}$  is given as

$$\begin{aligned} (h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) &= \dim_{\mathbb{C}} \text{Hom}(X, Y) - \dim_{\mathbb{C}} \text{Hom}(X, Y[1]) \\ &\quad + \dim_{\mathbb{C}} \text{Hom}(Y, X) - \dim_{\mathbb{C}} \text{Hom}(Y, X[1]) \end{aligned}$$

for any  $X \in \mathcal{P}_2(A, \mathbf{d}_1), Y \in \mathcal{P}_2(A, \mathbf{d}_2)$ . It is well-defined.

Let  $M(\mathcal{P}_2(A, \mathbf{d}))$  is the space of  $G_{\mathbf{d}}$ -invariant constructible functions over  $\mathcal{P}_2(A, \mathbf{d})$ . A function  $\hat{f} \in M(\mathcal{P}_2(A, \mathbf{d}))$  is called indecomposable if any point in  $\text{supp}(f)$  is indecomposable in  $\mathcal{K}_2(\mathcal{P}(A))$ . Let  $I_{GT}(\mathbf{d})$  be the  $\mathbb{C}$ -space of all indecomposable  $G_{\mathbf{d}}$ -invariant functions in  $M(\mathcal{P}_2(A, \mathbf{d}))$ . We set

$$\mathfrak{n} = \bigoplus_{\mathbf{d} \in K_0} I_{GT}(\mathbf{d})$$

and

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}.$$

We define the Lie bracket operation on  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$  by the following formulae:

$$[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}] = [\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathfrak{n}} + \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]}) h_{\mathbf{d}_1}$$

where  $\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]} = (\mathcal{O}_{\underline{e}''} \cap \mathcal{O}_{\underline{e}'}[1])/G_{\underline{e}''}$  and  $[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathfrak{n}}(L) := F_{\mathcal{O}_1\mathcal{O}_2}^L - F_{\mathcal{O}_2\mathcal{O}_1}^L$ ,

$$[h_{\mathbf{d}_1}, \hat{1}_{\mathcal{O}_2}] := (h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) \hat{1}_{\mathcal{O}_2}, [\hat{1}_{\mathcal{O}_2}, h_{\mathbf{d}_1}] = -(h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) \hat{1}_{\mathcal{O}_2},$$

$$[h_{\mathbf{d}_1}, h_{\mathbf{d}_2}] := 0.$$

For  $\hat{f} \in I_{GT}(\mathbf{d}_1)$  and  $\hat{g} \in I_{GT}(\mathbf{d}_2)$ , we can write the formulae in the following integral form:

$$\begin{aligned} [\hat{f}, \hat{g}]_{\mathfrak{n}}(x) &= \int_{V(\text{supp}(f), \text{supp}(g); x)} f(x')g(x'') - \int_{V(\text{supp}(g), \text{supp}(f); x)} g(x')f(x'') \\ [\hat{f}, h_{\mathbf{d}}] &= \int_{\text{supp}(f)} f(x)(h_{\mathbf{d}_1} | h_{\mathbf{d}}) \hat{1}_{f^{-1}(f(x))}. \end{aligned}$$

We will prove the following two of the main theorems in this paper.

**Theorem C** *Under the Lie bracket  $[-, -]$  defined as above,  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$  is a Lie algebra over  $\mathbb{C}$ .*



**Theorem D** *There exists a symmetric bilinear form  $(-|-)$  defined on  $\mathfrak{g}$  satisfying that*

- (a)  $(-|-)$  is invariant in the following sense:
  - (i)  $([h_{\mathbf{d}}, \hat{1}_{\mathcal{O}_1}] | \hat{1}_{\mathcal{O}_2}) = (h_{\mathbf{d}} | [\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]);$
  - (ii)  $([\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}] | \hat{1}_{\mathcal{O}_3}) = (\hat{1}_{\mathcal{O}_1} | [\hat{1}_{\mathcal{O}_2}, \hat{1}_{\mathcal{O}_3}]).$
- (b)  $(-|-)|_{\mathfrak{h}}$  is defined as above.
- (c)  $(\mathfrak{h} | \mathfrak{n}) = 0.$
- (d) for any  $\mathcal{O}_1, \mathcal{O}_2$  indecomposable

$$(\hat{1}_{\mathcal{O}_1} | \hat{1}_{\mathcal{O}_2}) = \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]}).$$

- (e)  $(-|-)|_{\mathfrak{n} \times \mathfrak{n}}$  is non-degenerated.

5.3. In this subsection, we compare the naïve Euler characteristics of quotient stacks induced by that defined in Section 5.1. We use the notations in Section 5.1. For fixed  $L \in \mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}$ , we set

$$\langle L \rangle := \{E \in \mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'} \mid F_{\mathcal{O}_1 \mathcal{O}_2}^E = F_{\mathcal{O}_2 \mathcal{O}_1}^E, F_{\mathcal{O}_2 \mathcal{O}_1}^E = F_{\mathcal{O}_1 \mathcal{O}_2}^E \text{ and } \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{LM} = \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{EM}\}.$$

In the same way as the proof of Proposition 3.15, we know  $\langle L \rangle$  is a constructible subset of  $\mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}$  and there exists a finite subset  $R(\underline{\mathbf{e}}'' + \underline{\mathbf{e}}')$  of  $\mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}$  such that

$$\mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'} = \bigcup_{L \in R(\underline{\mathbf{e}}'' + \underline{\mathbf{e}}')} \langle L \rangle.$$

Then by Lemma 3.8 and 3.9, we obtain the following result.

**Lemma 5.1.** *With the above notations, we have*

$$\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}M} = \sum_{L \in R(\underline{\mathbf{e}}'' + \underline{\mathbf{e}}')} \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{LM} \cdot \chi(\langle L \rangle / G_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}).$$

Dually, for fixed  $L' \in \mathcal{L}_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'}$ , we set

$$\langle L' \rangle^* := \{E \in \mathcal{L}_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'} \mid F_{\mathcal{O}_2 \mathcal{O}}^E = F_{\mathcal{O}_2 \mathcal{O}}^{L'}, F_{\mathcal{O} \mathcal{O}_2}^E = F_{\mathcal{O} \mathcal{O}_2}^{L'} \text{ and } \chi_{\mathcal{O}_1(\mathcal{O}_2 \mathcal{O})}^{ML'} = \chi_{\mathcal{O}_1(\mathcal{O}_2 \mathcal{O})}^{ME}\}.$$

Then there exists a finite subset  $R^*(\underline{\mathbf{e}} + \underline{\mathbf{e}}')$  of  $\mathcal{L}'_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'}$  such that

$$\mathcal{L}'_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'} = \bigcup_{L' \in R^*(\underline{\mathbf{e}} + \underline{\mathbf{e}}')} \langle L' \rangle^*.$$

**Lemma 5.2.** *With the above notations, we have*

$$\chi_{\mathcal{O}_1(\mathcal{O}_2 \mathcal{O})}^{M\mathcal{L}'} = \sum_{L' \in R^*(\underline{\mathbf{e}} + \underline{\mathbf{e}}')} \chi_{\mathcal{O}_1(\mathcal{O}_2 \mathcal{O})}^{ML'} \cdot \chi(\langle L' \rangle^* / G_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'}).$$

**Proposition 5.3.** If  $\mathcal{O}_1, \mathcal{O}_2$  are indecomposable, then

$$\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}M} = \chi_{\mathcal{O}_1(\mathcal{O}_2 \mathcal{O})}^{M\mathcal{L}'}.$$

Our proof follows [PX3] and some improvements in [Hu].

*Proof.* First we construct a map:

$$W(\mathcal{O}_{1\underline{\mathbf{e}}''}, \mathcal{O}_{2\underline{\mathbf{e}}'}; \mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}) \times W(\mathcal{L}_{\underline{\mathbf{e}}'' + \underline{\mathbf{e}}'}, \mathcal{O}_{\underline{\mathbf{e}}}; M) \xrightarrow{\tau} W(\mathcal{O}_{1\underline{\mathbf{e}}''}, \mathcal{L}'_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'}; M) \times W(\mathcal{O}_{2\underline{\mathbf{e}}'}, \mathcal{O}_{\underline{\mathbf{e}}}; \mathcal{L}'_{\underline{\mathbf{e}} + \underline{\mathbf{e}}'})$$

sending  $((f, g, h), (l, m, n))$  to  $((f', g', h'), (l', m', n'))$  via the following commutative diagram:

$$(5.1) \quad \begin{array}{ccccccc} Z & \xlongequal{\quad} & Z & & & & \\ \downarrow l' & & \downarrow l & & & & \\ L' & \xrightarrow{\quad f' \quad} & M & \xrightarrow{\quad g' \quad} & X & \xrightarrow{\quad h' \quad} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ Y & \xrightarrow{\quad f \quad} & L & \xrightarrow{\quad g \quad} & X & \xrightarrow{\quad h \quad} & Y[1] \\ \downarrow n' & & \downarrow n & & & & \\ Z[1] & \xlongequal{\quad} & Z[1] & & & & \end{array}$$

where  $L'$  is the direct summand of  $\text{Cone}((m, f)[-1])$  such that the complement is a contractible complex and  $L' \in \mathcal{L}'_{\mathbf{e}+\mathbf{e}'}$  and we denote the natural embedding  $i : L' \rightarrow \text{Cone}((m, f)[-1])$ . Hence  $f' = (0, 1, 0)^T \circ i$  and  $m' = (0, 0, 1)^T \circ i$ , where  $(0, 1, 0)^T$  and  $(0, 0, 1)^T$  are the natural projections:

$$\text{Cone}((m, f)[-1]) \rightarrow M \oplus Y \rightarrow M \text{ and } \text{Cone}((m, f)[-1]) \rightarrow M \oplus Y \rightarrow Y.$$

Using the property of cone in the triangulated category,  $g', h', l', n'$  can be expressed by the composition of  $f, g, h, l, m, n$  and some quasi-isomorphisms, by Lemma 2.1, 2.2 and 2.3, we deduce that these expressions are algebraic. Hence,  $\tau$  is a morphism of varieties.

Moreover, we claim that  $\tau$  is invariant under group action. For  $(a, b, c, d) \in G_{\mathbf{e}''} \times G_{\mathbf{e}'} \times G_{\mathbf{e}} \times G_{\mathbf{e}}$ , considering the pair  $((\bar{f}, \bar{g}, \bar{h}), (\bar{l}, \bar{m}, \bar{n})) = (a, b, c, d) \circ ((f, g, h), (l, m, n))$ .

$$\begin{array}{ccccccc} Y & \xrightarrow{f} & L & \xrightarrow{g} & X & \xrightarrow{h} & Y[1] \\ \downarrow c & & \downarrow b & & \downarrow a & & \downarrow c[1] \\ Y^* & \xrightarrow{\bar{f}} & L^* & \xrightarrow{\bar{g}} & X^* & \xrightarrow{\bar{h}} & Y^*[1] \end{array} \quad \begin{array}{ccccccc} Z & \xrightarrow{l} & M & \xrightarrow{m} & L & \xrightarrow{n} & Z[1] \\ \downarrow d & & \parallel & & \downarrow b & & \downarrow d[1] \\ Z^* & \xrightarrow{\bar{l}} & M & \xrightarrow{\bar{m}} & L^* & \xrightarrow{\bar{n}} & Z^*[1] \end{array}$$

Using the similar construction as diagram (5.1), we again find some  $L'' \in \mathcal{L}'_{\mathbf{e}+\mathbf{e}'}$  and a pair  $((\bar{f}', \bar{g}', \bar{h}'), (\bar{l}', \bar{m}', \bar{n}')) \in W(X^*, L''; M) \times W(Y^*, Z^*; L'')$  satisfying:

$$\tau((\bar{f}, \bar{g}, \bar{h}), (\bar{l}, \bar{m}, \bar{n})) = ((\bar{f}', \bar{g}', \bar{h}'), (\bar{l}', \bar{m}', \bar{n}'))$$

Following the construction of  $L'$ , there exists  $b'$  such that the following diagram of exact triangles commutes:

$$\begin{array}{ccccccc} L' & \xrightarrow{(f' - m')^t} & M \oplus Y & \xrightarrow{(m, f)} & L & \xrightarrow{h'g} & L'[1] \\ \downarrow b' & & \downarrow \begin{pmatrix} 1 & c \end{pmatrix} & & \downarrow b & & \downarrow b'[1] \\ L'' & \xrightarrow{(\bar{f}' - \bar{m}')^t} & M \oplus Y^* & \xrightarrow{(\bar{m}, \bar{f})} & L^* & \xrightarrow{\bar{h}'\bar{g}} & L''[1] \end{array}$$

It follows that  $b'$  is an isomorphism, and  $b' \in G_{\mathbf{e}+\mathbf{e}'}$ .

Similarly there exist  $a'$  and  $c'$  giving the following commutative diagram of exact triangles:

$$\begin{array}{ccccccc} L' & \xrightarrow{f'} & M & \xrightarrow{g'} & X & \xrightarrow{h'} & L'[1] \\ \downarrow b' & & \parallel & & \downarrow a' & & \downarrow b'[1] \\ L'' & \xrightarrow{\bar{f}'} & M & \xrightarrow{\bar{g}'} & X^* & \xrightarrow{\bar{h}'} & L''[1] \end{array} \quad \begin{array}{ccccccc} Z & \xrightarrow{l'} & L' & \xrightarrow{m'} & Y & \xrightarrow{n'} & Z[1] \\ \downarrow d' & & \downarrow b' & & \downarrow c & & \downarrow d'[1] \\ Z^* & \xrightarrow{\bar{l}'} & L' & \xrightarrow{\bar{m}'} & Y^* & \xrightarrow{\bar{n}'} & Z^*[1] \end{array}$$

This shows  $((\bar{f}', \bar{g}', \bar{h}'), (\bar{l}', \bar{m}', \bar{n}')) = (a', b', c, d') \circ ((f', g', h'), (l', m', n'))$  and so the two pairs of exact triangles lie in the same orbit. Hence, The morphism  $\tau$  induces the following 1-morphism:

$$(W(\mathcal{O}_1, \mathcal{O}_2; \mathcal{L}) \times W(\mathcal{L}, \mathcal{O}; M))^\wedge \xrightarrow{\tau^\wedge} (W(\mathcal{O}_1, \mathcal{L}'; M) \times W(\mathcal{O}_2, \mathcal{O}; \mathcal{L}'))^\wedge$$

Then we have a pseudomorphism

$$(W(\mathcal{O}_1, \mathcal{O}_2; \mathcal{L}) \times W(\mathcal{L}, \mathcal{O}; M))^\wedge(\mathbb{C}) \xrightarrow{(\tau^\wedge)_*} (W(\mathcal{O}_1, \mathcal{L}'; M) \times W(\mathcal{O}_2, \mathcal{O}; \mathcal{L}'))^\wedge(\mathbb{C}).$$

Depending on the symmetry of the construction of  $L'$ , we can construct the inverse of  $(\tau^\wedge)_*$ . Therefore,  $(\tau^\wedge)_*$  is a pseudoisomorphism. The proof is completed.  $\square$

We introduce some notations. Let

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} := \bigcup_{L \in \mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}'} W(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}'}; L) \times W(L, \mathcal{O}_{\underline{\mathbf{e}}}; M)$$

and let  $c(L) = ((f, g, h)(l, m, n))$  denotes the triangles as follows:

$$(5.2) \quad Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1], \quad Z \xrightarrow{l} M \xrightarrow{m} L \xrightarrow{n} Z[1].$$

We define

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M}(1) = \{c(L) \in W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} \mid L \not\cong M \oplus Z[1] \text{ for any } Z \in \mathcal{O}_{\underline{\mathbf{e}}}\},$$

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M}(2) = \{c(L) \in W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} \mid L \cong M \oplus Z[1], L \not\cong X \oplus Y \text{ and } X \not\cong Y\},$$

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M}(3) = \{c(L) \in W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} \mid L \cong M \oplus Z[1], L \cong X \oplus Y \text{ and } X \not\cong Y\},$$

and

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M}(4) = \{c(L) \in W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} \mid L \cong M \oplus Z[1], L \cong X \oplus Y \text{ and } X \cong Y\}.$$

It is clear that

$$W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M} = \bigcup_{i=1}^4 W_{(\mathcal{O}_{\underline{\mathbf{e}}}'', \mathcal{O}_{\underline{\mathbf{e}}}') \mathcal{O}_{\underline{\mathbf{e}}}}^{\mathcal{L}_{\underline{\mathbf{e}}}' + \underline{\mathbf{e}}' M}(i).$$

The above partition induces the partitions of  $(W_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M})^\wedge$ . Hence,

$$\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M} = \sum_{i=1}^4 \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M}(i)$$

where  $\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M}(i) = \chi((W_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M})^\wedge(i))$ .

Define

$$V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L} M} = \bigcup_{L \in R(\underline{\mathbf{e}}'' + \underline{\mathbf{e}}')} V(\mathcal{O}_1, \mathcal{O}_2; L) \times V(\langle L \rangle, \mathcal{O}; M)$$

Put  $\langle L \rangle_0 = \{E \in \langle L \rangle \mid E \cong M \oplus Z[1] \text{ for some } Z \in \mathcal{O}_{\underline{e}}\}$  and  $\langle L \rangle_1 = \langle L \rangle \setminus \langle L \rangle_0$ . Hence,

$$V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M} = V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(0) \bigcup V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(1)$$

where  $V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(i) = \bigcup_{L \in R(\underline{e}'' + \underline{e}')} V(\mathcal{O}_1, \mathcal{O}_2; L) \times V(\langle L \rangle_i, \mathcal{O}; M)$  for  $i = 0, 1$ .

**Lemma 5.4.** *For fixed  $M$  and  $L \in \mathcal{L}_{\underline{e}'' + \underline{e}'}$ , we have*

$$\chi(V(\mathcal{O}_1, \mathcal{O}_2; L) \times V(\mathcal{O}, \langle L \rangle_1; M)) = \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\langle L \rangle_1 M}.$$

*Proof.* Consider the  $G_{\underline{e}'' + \underline{e}'}$ -action

$$W(\mathcal{O}_{\underline{e}}, \langle L \rangle_1; M)^* \rightarrow V(\mathcal{O}, \langle L \rangle_1; M)$$

For any  $(l, m, n)^*$ , the stable subgroup denoted by  $B(n)$  is isomorphic to the affine space  $\text{Hom}(Z[1], M)n$ . Then we have a 1-morphism

$$p : (W_{(\mathcal{O}_{\underline{e}''} \mathcal{O}_{\underline{e}'}) \mathcal{O}_{\underline{e}}}^{\langle L \rangle_1 M})^\wedge \rightarrow V(\mathcal{O}, \langle L \rangle_1; M).$$

It induces the pseudomorphism

$$p_* : (W_{(\mathcal{O}_{\underline{e}''} \mathcal{O}_{\underline{e}'}) \mathcal{O}_{\underline{e}}}^{\langle L \rangle_1 M})^\wedge(\mathbb{C}) \rightarrow V(\mathcal{O}, \langle L \rangle_1; M)(\mathbb{C}).$$

For any  $(l, m, n)^\wedge \in V(\mathcal{O}, \langle L \rangle_1; M)$ , the fibre  $p_*^{-1}(l, m, n)$  is pseudomorphic to  $[V(\mathcal{O}_{1\underline{e}'}, \mathcal{O}_{2\underline{e}'}; E)/B(n)](\mathbb{C})$  where  $E$  occurs in the triangle  $(l, m, n)$  as diagram (5.2). Under the action of  $B(n)$ , the stable subgroup for any  $(f, g, h) \in V(\mathcal{O}_{1\underline{e}'}, \mathcal{O}_{2\underline{e}'}; \langle L \rangle_1)$  is

$$\{b \in B(n) \mid fa' = bf \text{ for some } a' \in \text{End} Y\}$$

which is the subspace of  $\text{Hom}(Z[1], L)n$ . By Proposition 3.4, Lemma 3.9 and 3.8, according to the fact  $F_{\mathcal{O}_1 \mathcal{O}_2}^E = F_{\mathcal{O}_1 \mathcal{O}_2}^L$  for any  $E \in \langle L \rangle_1$ , we have

$$\chi(V(\mathcal{O}_1, \mathcal{O}_2; L) \times V(\mathcal{O}, \langle L \rangle_1; M)) = \chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\langle L \rangle_1 M}.$$

□

The Lemma naturally induces the following Proposition.

**Proposition 5.5.** For fixed  $M$ , we have

$$\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(1) = V_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(1).$$

The following Lemma is a natural corollary of Proposition 3.4.

**Lemma 5.6.** *Let  $X, Y \in \mathcal{P}_2(A)$  be two indecomposable objects, then*

$$\chi(\text{Hom}_{\mathcal{P}_2(A)}(X, Y)) = 1 \text{ and } \chi(\text{Aut}_{\mathcal{P}_2(A)} X) = 0.$$

**Proposition 5.7.** Let  $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}$  be indecomposable as above.

- (I) If  $L \cong M \oplus Z[1]$  for some  $Z \in \mathcal{O}$  and  $L \notin \mathcal{O}_1 \oplus \mathcal{O}_2$ , then  $F_{\mathcal{O}_1 \mathcal{O}_2}^L = 0$  and  $\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(2) = \chi_{\mathcal{O}(\mathcal{O}_1 \mathcal{O}_2)}^{\mathcal{L}^M}(2)$ .
- (II) If  $X \in \mathcal{O}_1, Y \in \mathcal{O}_2$  such that  $X \not\cong Y$ , then  $\chi_{(XY)Y[1]}^{X \oplus Y, X} = \chi_{X[1](XY)}^{Y, X \oplus Y} = 1$ ,  $\chi_{(XY)X[1]}^{X \oplus Y, Y} - 1 = \chi_{Y[1](XY)}^{X, X \oplus Y} - 1 = \dim_{\mathbb{C}} \text{Hom}(Y, X)$ .
- (III) If  $X$  is indecomposable, then  $\chi_{(XX)X[1]}^{X \oplus X, X} = \chi_{X[1](XX)}^{X, X \oplus X}$ . Hence,  $\chi_{(\mathcal{O}_1 \mathcal{O}_2) \mathcal{O}}^{\mathcal{L}^M}(4) = \chi_{\mathcal{O}(\mathcal{O}_1 \mathcal{O}_2)}^{\mathcal{L}^M}(4)$ .

*Proof.* Let  $L \in M \oplus \mathcal{O}[1]$ . Then  $L \cong M \oplus Z[1]$  for some  $Z \in \mathcal{O}$ . Define

$$\phi : V(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1]) \rightarrow ((W(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1]) \times W(M \oplus Z[1], Z; M))^\wedge$$

mapping  $(f, g, h)^\wedge$  to  $((f, g, h), (0, (1, 0)^t, (0, 1)))^\wedge$ . It is a surjective morphism. The group action  $W(M \oplus Z[1], Z; M)^* \rightarrow V(M \oplus Z[1], Z; M)$  has the stable subgroup  $B(n)$ . It is independent of  $n$  since  $V(M \oplus Z[1], Z; M)$  has only one element. We denote it by  $B$ . It is given as follows.

$$B = \{b = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix} \in \text{End}(M \oplus Z[1]) \mid b_1 \in \text{Hom}(Z[1], M), b_2 \in \text{Aut} Z[1]\}$$

Then  $\text{Stab}((f, g, h)^\wedge) = \{b \in B \mid fc = bf \text{ for some } c \in \text{Aut} Y\}$ .

For the case  $L \notin \mathcal{O}_1 \oplus \mathcal{O}_2$ , then

$$1 - \text{Stab}((f, g, h)^\wedge) = \{b' = \begin{pmatrix} 0 & 0 \\ b'_1 & b'_2 \end{pmatrix} \mid b'_1 \in \text{Hom}(Z[1], M), b'_2 \in \text{radEnd} Z[1]$$

such that  $fc = b'f$  for some  $c \in \text{End} Y\}$

(see Section 7.3 of [PX3]). It is an affine space.

For any  $x \in ((W(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1]) \times W(M \oplus Z[1], Z; M))^\wedge$ , by Lemma 3.9 and Lemma 5.4,

$$\chi(\phi^{-1}(x)) = \chi(B) = \chi(\text{Hom}(Z[1], M)) \cdot \chi(\text{Aut} Z[1]) = 0$$

and clearly  $F_{M \oplus Z[1], Z}^M = 1$ . Therefore

$$F_{\mathcal{O}_1 \mathcal{O}_2}^{M \oplus Z[1]} = \chi(V(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1])) = 0.$$

The maps  $\phi$  and  $\varphi$  induce a homeomorphism:

$$\begin{aligned} ((W(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1]) \times W(Z, M \oplus Z[1]; M))^\wedge \\ \rightarrow ((W(\mathcal{O}_1, \mathcal{O}_2; M \oplus Z[1]) \times W(M \oplus Z[1], Z; M))^\wedge \end{aligned}$$

Hence,  $\chi_{(\mathcal{O}_1 \mathcal{O}_2)Z}^{M \oplus Z[1], M} = \chi_{Z(\mathcal{O}_1 \mathcal{O}_2)}^{M, M \oplus Z[1]}$ . The statement (I) is proved. The statement (III) can be proved by the similar discussion.

For the case  $L \in \mathcal{O}_1 \oplus \mathcal{O}_2$ , then  $L \cong X \oplus Y$  for some  $X \in \mathcal{O}_1, Y \in \mathcal{O}_2$  and  $X \not\cong Y$ . Every orbit in  $V(X, Y, X \oplus Y)$  is of the form as  $((\begin{pmatrix} \theta_1 \\ 1 \end{pmatrix}, (1, \theta_2), 0)^\wedge$  such that  $\theta_1, \theta_2 : Y \rightarrow X, \theta_1 + \theta_2 = 0$ . So  $V(X, Y; X \oplus Y) \cong \text{Hom}(Y, X)$ . This induces the maps:

$$\text{Hom}(Y, X) \rightarrow (W(X, Y; X \oplus Y) \times W(X \oplus Y, Y[1]; X))^\wedge$$

and

$$\text{Hom}(Y, X) \rightarrow (W(X, Y; X \oplus Y) \times W(X \oplus Y, X[1]; Y))^\wedge$$

The set  $(W(X, Y; X \oplus Y) \times W(X \oplus Y, Y[1]; X))^\wedge$  has unique element, implying  $\chi_{(XY)Y[1]}^{X \oplus Y, X} = 1$ . The natural action of group  $\text{Aut} X$  on  $\text{Hom}(Y, X)$  is free except on the point 0. It induces a homeomorphism:

$$(W(X, Y; X \oplus Y) \times W(X \oplus Y, X[1]; Y))^\wedge \setminus \{0\} \cong (\text{Hom}(Y, X) \setminus \{0\})/\text{Aut} X$$

So  $\chi_{(XY)X[1]}^{X \oplus Y, Y} = 1 + \chi(\text{Hom}(Y, X) \setminus \{0\}/\text{Aut} X)$ . As we know (see [Rie]),  $\text{Aut} X$  is the direct product of  $\mathbb{C}^*$  and the subgroup  $1 + \text{radEnd} X$  which is contractible. Hence,

$$\chi(\text{Hom}(Y, X) \setminus \{0\}/\text{Aut} X) = \chi(\text{Hom}(Y, X) \setminus \{0\}/\mathbb{C}^*) = \chi(\mathbb{C}\mathbb{P}^{\dim_{\mathbb{C}} \text{Hom}(Y, X)})$$

$$= \dim_{\mathbb{C}} \text{Hom}(Y, X).$$

The statement (II) is proved.  $\square$

**Lemma 5.8.** *If  $L$  is decomposable or 0, then  $F_{\mathcal{O}_1\mathcal{O}_2}^L - F_{\mathcal{O}_2\mathcal{O}_1}^L = 0$ .*

*Proof.* If  $L$  is 0, it is trivial. If  $L$  is decomposable, we assume  $L = M \oplus Z[1]$  with  $Z$  indecomposable and  $M \not\cong 0$ . If  $L \not\cong X \oplus Y$  for any  $X \in \mathcal{O}_1, Y \in \mathcal{O}_2$ , then by Proposition 5.5,  $F_{\mathcal{O}_1\mathcal{O}_2}^L = F_{\mathcal{O}_1\mathcal{O}_2}^L = 0$ . Using this fact, we have:

$$F_{\mathcal{O}_1\mathcal{O}_2}^L - F_{\mathcal{O}_2\mathcal{O}_1}^L = (F_{MZ[1]}^{M \oplus Z[1]} - F_{Z[1]M}^{M \oplus Z[1]})(1_{\mathcal{O}_1}(M)1_{\mathcal{O}_2}(Z[1]) - 1_{\mathcal{O}_1}(Z[1])1_{\mathcal{O}_2}(M))$$

So we can assume that both  $M, Z[1]$  are indecomposable and  $M \not\cong Z[1]$ . By the proof of II of Proposition 5.7,  $F_{MZ[1]}^{M \oplus Z[1]} = \chi(\text{Hom}(Z[1], M)) = 1$ ,  $F_{Z[1]M}^{M \oplus Z[1]} = \chi(\text{Hom}(M, Z[1])) = 1$ .  $\square$

**Corollary 5.9.** For fixed  $M$ , we have

$$\chi(V_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}(0)) - \chi(V_{(\mathcal{O}_2\mathcal{O}_1)\mathcal{O}}^{\mathcal{LM}}(0)) = 0.$$

and

$$\chi(V_{\mathcal{O}(\mathcal{O}_1\mathcal{O}_2)}^{\mathcal{LM}}(0)) - \chi(V_{\mathcal{O}(\mathcal{O}_2\mathcal{O}_1)}^{\mathcal{LM}}(0)) = 0.$$

We note that we can translate the discussion in this section for  $W_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}'}$  and  $V_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}'}$ . the discussion is completely dual.

5.4. Now, we come to prove Theorem C and D.

**Proof of Theorem C:** It suffices to verify the Jacobi identity:

$$\Delta = [[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}], \hat{1}_{\mathcal{O}}] - [[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}}], \hat{1}_{\mathcal{O}_2}] - [[\hat{1}_{\mathcal{O}}, \hat{1}_{\mathcal{O}_2}], \hat{1}_{\mathcal{O}_1}] = 0$$

$$\Delta' = [[h_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}], \hat{1}_{\mathcal{O}}] - [[h_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}}], \hat{1}_{\mathcal{O}_2}] - [[\hat{1}_{\mathcal{O}}, \hat{1}_{\mathcal{O}_2}], h_{\mathcal{O}_1}] = 0$$

$$\Delta'' = [[h_{\mathcal{O}_1}, h_{\mathcal{O}_2}], \hat{1}_{\mathcal{O}}] - [[h_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}}], h_{\mathcal{O}_2}] - [[\hat{1}_{\mathcal{O}}, h_{\mathcal{O}_2}], h_{\mathcal{O}_1}] = 0$$

We will follow tightly [PX3] and [Hu]. First,  $\Delta$  (also  $\Delta'$  and  $\Delta''$ ) is a  $G_{\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}}$ -invariant constructible function by discussion in Section 3.4. Moreover, according to the definition of Lie bracket operation in Section 5.2 and the discussion in Section 5.3, we know

$$(\hat{1}_{\mathcal{O}_1} * \hat{1}_{\mathcal{O}_2}) * \hat{1}_{\mathcal{O}}(M) = \sum_{L \in R(\mathbf{e}'' + \mathbf{e}')} F_{\mathcal{O}_1\mathcal{O}_2}^L \cdot F_{\langle L \rangle \mathcal{O}_1}^M$$

for  $M \in \mathcal{P}_2(A, \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d})$  and

$$[[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}], \hat{1}_{\mathcal{O}}] = [[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathbf{n}}, \hat{1}_{\mathcal{O}}]_{\mathbf{n}} - \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]})(h_{\mathbf{d}_1} \mid h_{\mathbf{d}}) \hat{1}_{\mathcal{O}} - (F_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{O}[1]} - F_{\mathcal{O}_2\mathcal{O}_1}^{\mathcal{O}[1]}) h_{\mathbf{d}}$$

where

$$[[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathbf{n}}, \hat{1}_{\mathcal{O}}]_{\mathbf{n}}(M) = \sum_{L \in R(\mathbf{e}'' + \mathbf{e}')} (F_{\mathcal{O}_1\mathcal{O}_2}^L - F_{\mathcal{O}_2\mathcal{O}_1}^L) \cdot (F_{\langle L \rangle \mathcal{O}_1}^M - F_{\mathcal{O}_1 \langle L \rangle}^M)$$

and

$$F_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{O}[1]} = \sum_{L \in R(\mathbf{e}'' + \mathbf{e}')} \chi(\langle L \rangle \cap \mathcal{O}[1]) F_{\mathcal{O}_1\mathcal{O}_2}^L.$$

Let us write

$$c_M := \Delta_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}^M + \Delta_{\mathcal{O}_2\mathcal{O}\mathcal{O}_1}^M + \Delta_{\mathcal{O}\mathcal{O}_1\mathcal{O}_2}^M - \Delta_{\mathcal{O}_2\mathcal{O}_1\mathcal{O}}^M - \Delta_{\mathcal{O}\mathcal{O}_2\mathcal{O}_1}^M - \Delta_{\mathcal{O}_1\mathcal{O}\mathcal{O}_2}^M,$$

where

$$\Delta_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}^M := \chi(V_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}) - \chi(V_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}}).$$

Define

$$b_M := (\chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]})(h_{\mathcal{O}_1} \mid h_{\mathcal{O}})\hat{1}_{\mathcal{O}} - \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}[1]})(h_{\mathcal{O}_1} \mid h_{\mathcal{O}_2})\hat{1}_{\mathcal{O}_2} - \chi(\overline{\mathcal{O} \cap \mathcal{O}_2[1]})(h_{\mathcal{O}} \mid h_{\mathcal{O}_1})\hat{1}_{\mathcal{O}_1})(M).$$

Set  $\gamma_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{L}} = F_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{L}} - F_{\mathcal{O}_2\mathcal{O}_1}^{\mathcal{L}}$ , we define

$$a_M := \gamma_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{O}[1]} h_{\mathcal{O}} - \gamma_{\mathcal{O}_1\mathcal{O}}^{\mathcal{O}_2[1]} h_{\mathcal{O}_2} - \gamma_{\mathcal{O}\mathcal{O}_2}^{\mathcal{O}_1[1]} h_{\mathcal{O}_1}$$

hence,

$$\Delta(M) = c_M - b_M - a_M$$

(1)  $c_M = c_M^1 + c_M^2$  and  $c_M^1 = 0$

By Proposition 5.3 and 5.7, we have

$$\Delta_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}^M = \chi(V_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}(0)) - \chi(V_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{\mathcal{LM}}(0)) + \sum_{i=2}^4 (\chi_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}'}(i) - \chi_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}(i)).$$

Set

$$\begin{aligned} c_M^1 &= (\chi(V_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}(0)) - \chi(V_{(\mathcal{O}_2\mathcal{O}_1)\mathcal{O}}^{\mathcal{LM}}(0))) - (\chi(V_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}}(0)) - \chi(V_{\mathcal{O}_1(\mathcal{O}\mathcal{O}_2)}^{M\mathcal{L}}(0))) \\ &\quad + (\chi(V_{(\mathcal{O}_2\mathcal{O})\mathcal{O}_1}^{\mathcal{LM}}(0)) - \chi(V_{(\mathcal{O}\mathcal{O}_2)\mathcal{O}_1}^{\mathcal{LM}}(0))) - (\chi(V_{\mathcal{O}_2(\mathcal{O}\mathcal{O}_1)}^{M\mathcal{L}}(0)) - \chi(V_{(\mathcal{O}_2\mathcal{O}_1)\mathcal{O}}^{M\mathcal{L}}(0))) \\ &\quad + (\chi(V_{(\mathcal{O}\mathcal{O}_1)\mathcal{O}_2}^{\mathcal{LM}}(0)) - \chi(V_{(\mathcal{O}_1\mathcal{O})\mathcal{O}_2}^{M\mathcal{L}}(0))) - (\chi(V_{\mathcal{O}(\mathcal{O}_1\mathcal{O}_2)}^{M\mathcal{L}}(0)) - \chi(V_{(\mathcal{O}\mathcal{O}_2)\mathcal{O}_1}^{M\mathcal{L}}(0))) \\ c_M^2 &= \sum_{i=2}^4 \{ -\chi_{(\mathcal{O}_1\mathcal{O}_2)\mathcal{O}}^{\mathcal{LM}}(i) + \chi_{\mathcal{O}_1(\mathcal{O}_2\mathcal{O})}^{M\mathcal{L}'}(i) - \chi_{(\mathcal{O}_2\mathcal{O})\mathcal{O}_1}^{\mathcal{LM}}(i) + \chi_{\mathcal{O}_2(\mathcal{O}\mathcal{O}_1)}^{M\mathcal{L}'}(i) - \chi_{(\mathcal{O}\mathcal{O}_1)\mathcal{O}_2}^{\mathcal{LM}}(i) \\ &\quad + \chi_{\mathcal{O}(\mathcal{O}_1\mathcal{O}_2)}^{M\mathcal{L}'}(i) + \chi_{(\mathcal{O}_2\mathcal{O}_1)\mathcal{O}}^{\mathcal{LM}}(i) - \chi_{\mathcal{O}_2(\mathcal{O}_1\mathcal{O})}^{M\mathcal{L}'}(i) + \chi_{(\mathcal{O}\mathcal{O}_2)\mathcal{O}_1}^{\mathcal{LM}}(i) - \chi_{\mathcal{O}(\mathcal{O}_2\mathcal{O}_1)}^{M\mathcal{L}'}(i) \\ &\quad + \chi_{(\mathcal{O}_1\mathcal{O})\mathcal{O}_2}^{\mathcal{LM}}(i) - \chi_{\mathcal{O}_1(\mathcal{O}\mathcal{O}_2)}^{M\mathcal{L}'}(i) \}. \end{aligned}$$

Then  $c_M = c_M^1 + c_M^2$ .

By Corollary 5.9,  $c_M^1 = 0$  and for  $c_M^2$ , we first remark the following fact (see Proposition 5.7):

$$\begin{aligned} &\sum_{t=2}^4 \chi_{(\mathcal{O}_i\mathcal{O}_j)\mathcal{O}_k}^{\mathcal{LM}}(t) - \chi_{\mathcal{O}_k(\mathcal{O}_i\mathcal{O}_j)}^{M\mathcal{L}'}(t) = \chi_{(\mathcal{O}_i\mathcal{O}_j)\mathcal{O}_k}^{\mathcal{LM}}(3) - \chi_{\mathcal{O}_k(\mathcal{O}_i\mathcal{O}_j)}^{M\mathcal{L}'}(3) \\ &= \int_{Z \in \overline{\mathcal{O}_i[1] \cap \mathcal{O}_k}} \dim_{\mathbb{C}} \text{Hom}(M, Z[1]) 1_{\mathcal{O}_j}(M) - \int_{Z \in \overline{\mathcal{O}_j[1] \cap \mathcal{O}_k}} \dim_{\mathbb{C}} \text{Hom}(Z[1], M) 1_{\mathcal{O}_i}(M) \end{aligned}$$

for any  $Z \in \mathcal{O}_k$ . We use this fact to substitute the terms in  $c_M^2$ , then

$$c_M^2 = b_M.$$

(2)  $a_M = 0$

We claim that for indecomposable constructible sets  $\mathcal{O}_i, \mathcal{O}_j$  and  $\mathcal{O}_k$ ,

$$(5.3) \quad F_{\mathcal{O}_i\mathcal{O}_j}^{\mathcal{O}_k[1]} = F_{\mathcal{O}_j\mathcal{O}_k}^{\mathcal{O}_i[1]}.$$

Without loss of generality, we assume that  $\mathcal{O}_s$  is an indecomposable constructible subset of  $\mathcal{P}_2(A, \underline{\mathbf{e}}_s)$  for  $s = i, j, k$ . Let  $X_i, X_j$  and  $X_k$  be any objects in them, respectively and the corresponding orbits are denoted by  $\mathcal{O}_{X_l}$  for  $l = i, j, k$ . Considering

group action (c) in Section 5.1, we find as the orbit spaces of  $V(X_i, X_j; X_k[1])$  and  $V(X_j[1], X_k[1]; X_i)$

$$\overline{V(X_i, X_j; X_k[1])} \cong \overline{V(X_j[1], X_k[1]; X_i)}$$

and the corresponding fibres are:

$$\{(bf, gb^{-1}, h)^\wedge \mid b \in G_{\underline{e}}\} \text{ and } \{(ag, ha^{-1}, f[1])^\wedge \mid a \in G_{\underline{e}''}\}$$

Depending on indecomposable property of  $X_i, X_j$  and  $X_k$ ,  $\text{Aut} X_i$  and  $\text{Aut} X_k$  can be decomposed as direct products of  $\mathbb{C}^*$  and contractible subgroups, hence, we can regard  $a, b \in \mathbb{C}^*$  in the above fibres so that two fibres have the same Euler characteristic. This shows  $F_{\mathcal{O}_{X_i} \mathcal{O}_{X_j}}^{X_k[1]} = F_{\mathcal{O}_{X_j} \mathcal{O}_{X_k}}^{X_i[1]}$ . Hence, by Lemma 3.9, we know

$$F_{\mathcal{O}_{X_i} \mathcal{O}_{X_j}}^{\mathcal{O}_{X_k}[1]} = F_{\mathcal{O}_{X_j} \mathcal{O}_{X_k}}^{\mathcal{O}_{X_i}[1]}.$$

Let  $\langle X_i, X_j, X_k \rangle = \{(E_i, E_j, E_k) \in \mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k \mid F_{\mathcal{O}_{E_i} \mathcal{O}_{E_j}}^{\mathcal{O}_{E_k}[1]} = F_{\mathcal{O}_{X_i} \mathcal{O}_{X_j}}^{\mathcal{O}_{X_k}[1]}\}$ . Then

$$\langle X_i, X_j, X_k \rangle = \langle X_j, X_k, X_i \rangle.$$

Consider the projection

$$W(\mathcal{O}_i, \mathcal{O}_j; \mathcal{O}_k) \rightarrow \mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k.$$

It induces the map:

$$V(\mathcal{O}_i, \mathcal{O}_j; \mathcal{O}_k) \rightarrow \overline{\mathcal{O}_i} \times \overline{\mathcal{O}_j} \times \overline{\mathcal{O}_k}$$

where  $\overline{\mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k}$  is the quotient space of  $\mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k$  under the action of  $G_{\underline{e}_i} \times G_{\underline{e}_j} \times G_{\underline{e}_k}$ . In the same way as the proof of Proposition 3.15, using Lemma 3.9, we can prove that there exists a finite subset  $R$  of  $\mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k$  such that

$$\overline{\mathcal{O}_i \times \mathcal{O}_j \times \mathcal{O}_k} = \bigcup_{(X_i, X_j, X_k) \in R} \overline{\langle X_i, X_j, X_k \rangle}$$

and

$$F_{\mathcal{O}_i \mathcal{O}_j}^{\mathcal{O}_k[1]} = \sum_{(X_i, X_j, X_k) \in R} \chi(\overline{\langle X_i, X_j, X_k \rangle}) \cdot F_{\mathcal{O}_{X_i} \mathcal{O}_{X_j}}^{\mathcal{O}_{X_k}[1]}.$$

and

$$F_{\mathcal{O}_j \mathcal{O}_k}^{\mathcal{O}_i[1]} = \sum_{(X_i, X_j, X_k) \in R} \chi(\overline{\langle X_j, X_k, X_i \rangle}) \cdot F_{\mathcal{O}_{X_j} \mathcal{O}_{X_k}}^{\mathcal{O}_{X_i}[1]}.$$

This implies our claim.

If  $h_{\mathcal{O}_1} + h_{\mathcal{O}_2} + h_{\mathcal{O}_3} \neq 0$ , then any term in  $a_M$  vanishes. So we assume  $h_{\mathcal{O}_1} + h_{\mathcal{O}_2} + h_{\mathcal{O}_3} = 0$ , in this case,  $a_M = 0$  follows our claim. Now consider  $\Delta'$ ,

$$\begin{aligned} \Delta' &= [[h_{\mathcal{O}_1}, 1_{\mathcal{O}_2}], 1_{\mathcal{O}}] - [[h_{\mathcal{O}_1}, 1_{\mathcal{O}}], 1_{\mathcal{O}_2}] - [[1_{\mathcal{O}}, 1_{\mathcal{O}_2}], h_{\mathcal{O}_1}] \\ &= (h_{\mathcal{O}_1} \mid h_{\mathcal{O}_2} + h_{\mathcal{O}})[1_{\mathcal{O}_2}, 1_{\mathcal{O}}] + (h_{\mathcal{O}_1} \mid h_{\mathcal{O}_2} + h_{\mathcal{O}})[1_{\mathcal{O}}, 1_{\mathcal{O}_2}] = 0 \end{aligned}$$

Finally according to the definition of the Lie bracket, it is easy to prove  $\Delta'' = 0$ . Thus we complete the proof of Theorem C.  $\square$

**Proof of Theorem D:** We claim  $(, )|_{\mathfrak{n} \times \mathfrak{n}}$  is non degenerated. For any dimension vector  $\mathbf{d}$  and  $\hat{f} \in I_{GT}(\mathbf{d})$ , without loss of generality, we may assume  $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$  for  $i \neq j$ . If  $\hat{f} \neq 0$ , then there exists  $m_i \neq 0$ . We take any  $L \in \mathcal{O}_i$  and let  $\mathcal{O}_{L[1]}$  be the orbit of  $L[1]$ , then  $(\hat{f} \mid \hat{1}_{\mathcal{O}_{L[1]}}) = m_i \neq 0$ . So it remains to prove that the bilinear



form defined by (b), (c) and (d) is symmetric and satisfies the condition (a). For (i),

$$([h_{\mathbf{d}}, \hat{1}_{\mathcal{O}_1}] | \hat{1}_{\mathcal{O}_2}) = (h_{\mathbf{d}} | h_{\mathcal{O}_1})(\hat{1}_{\mathcal{O}_1} | \hat{1}_{\mathcal{O}_2}) = (h_{\mathbf{d}} | h_{\mathcal{O}_1})\chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]})$$

and

$$(h_{\mathbf{d}} | [\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]) = (h_{\mathbf{d}} | h_{\mathcal{O}_1 \cap \mathcal{O}_2[1]})\chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]}) = (h_{\mathbf{d}} | h_{\mathcal{O}_1})\chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]})$$

For (ii), we only need to prove the following identity:

$$([\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathbf{n}} | \hat{1}_{\mathcal{O}_3}) = (\hat{1}_{\mathcal{O}_1} | [\hat{1}_{\mathcal{O}_2}, \hat{1}_{\mathcal{O}_3}]_{\mathbf{n}})$$

We have known the Lie bracket of constructible functions is still a constructible function. Hence, we can set  $[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathbf{n}} = \sum m_i \hat{1}_{\mathcal{C}_i}$  and  $[\hat{1}_{\mathcal{O}_2}, \hat{1}_{\mathcal{O}_3}]_{\mathbf{n}} = \sum n_j \hat{1}_{\mathcal{D}_j}$ . For any  $L \in \mathcal{C}_i$  and  $M \in \mathcal{D}_j$ ,

$$m_i = F_{\mathcal{O}_1 \mathcal{O}_2}^L - F_{\mathcal{O}_2 \mathcal{O}_1}^L \text{ and } n_j = F_{\mathcal{O}_2 \mathcal{O}_3}^M - F_{\mathcal{O}_3 \mathcal{O}_2}^M.$$

Therefore,

$$\begin{aligned} ([\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathbf{n}} | \hat{1}_{\mathcal{O}_3}) &= \sum m_i \chi(\overline{\mathcal{C}_i \cap \mathcal{O}_3[1]}) = F_{\mathcal{O}_1 \mathcal{O}_2}^{\mathcal{O}_3[1]} - F_{\mathcal{O}_2 \mathcal{O}_1}^{\mathcal{O}_3[1]} \\ (\hat{1}_{\mathcal{O}_1} | [\hat{1}_{\mathcal{O}_2}, \hat{1}_{\mathcal{O}_3}]_{\mathbf{n}}) &= \sum n_j \chi(\overline{\mathcal{D}_j \cap \mathcal{O}_1[1]}) = F_{\mathcal{O}_2 \mathcal{O}_3}^{\mathcal{O}_1[1]} - F_{\mathcal{O}_3 \mathcal{O}_2}^{\mathcal{O}_1[1]} \end{aligned}$$

Using our claim (5.3) in the proof of Theorem C, we obtain the identity.  $\square$

5.5. Let  $K_0$  be the Grothendieck group of  $\mathcal{D}^b(A)$  and  $\mathfrak{h}_{\mathbb{Z}} = K_0$ . For  $\mathbf{d} \in K_0$ , let  $M_{GT}^{\mathbb{Z}}(\mathbf{d})$  and  $I_{GT}^{\mathbb{Z}}(\mathbf{d})$  be the set of all  $\hat{f} \in M(\mathcal{P}_2(A, \mathbf{d}))$  and  $\hat{f} \in I_{GT}(\mathbf{d})$  such that all values of  $\hat{f}$  are in  $\mathbb{Z}$ , respectively. We consider the  $\mathbb{Z}$ -spaces  $\mathbf{n}_{\mathbb{Z}} = \bigoplus_{\mathbf{d} \in K_0} I_{GT}^{\mathbb{Z}}(\mathbf{d})$  and  $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} \oplus \mathbf{n}_{\mathbb{Z}}$ . Then Theorem C and Theorem D have the following  $\mathbb{Z}$ -form.

**Theorem 5.10.** (1) *The  $\mathbb{Z}$ -space  $\mathfrak{g}_{\mathbb{Z}}$  with the bracket  $[-, -]$  is a Lie algebra.*

(2) *The symmetric bilinear form  $(-|-)$  is invariant over  $\mathfrak{g}_{\mathbb{Z}}$  in the sense*

$$([x, y]|z) = (x|[y, z]) \text{ for any } x, y \text{ and } z \in \mathfrak{g}_{\mathbb{Z}}.$$

(3)  *$(-|-)|_{\mathbf{n}_{\mathbb{Z}} \times \mathbf{n}_{\mathbb{Z}}}$  is non-degenerated.*

## 6. REALIZATION OF GENERALIZED KAC-MOODY LIE ALGEBRA

6.1. As an application, we consider the case  $A = \mathbb{C}Q$  where  $Q$  is a finite quiver without oriented cycle. In this section, we always use  $\mathfrak{g}$  to denote the Lie algebra arising from  $\mathcal{P}_2(A)$  which is defined in Section 5.2. We will show that the above Lie algebra is a generalized Kac-Moody Lie algebra and the corresponding symmetric Kac-Moody algebra is a subalgebra of it, by following the methods in [SV] and [DX]. When  $Q$  is a tame quiver, we will demonstrate that the precise structure of the symmetric affine Lie algebra can be revealed from the derived category of representations of  $Q$ . This phenomenon was first discovered in [FMV] and extended in [LP].

We fix the embedding  $\text{mod } A$  in  $D^b(A)$  by taking any  $X \in \text{mod } A$  as a stalk complex  $X^\bullet = (X_i)$  with  $X_0 = X$  and  $X_i = 0$  for  $i \neq 0$ . Hence, we have the induced embedding of  $\text{mod } A$  in  $\mathcal{D}_2(A)$  since the functor  $F$  in Section 4.1 is dense (See [PX2] Corollary 7.1 or [Ke]). Moreover,  $\mathcal{D}_2(A) \cong \mathcal{K}_2(\mathcal{P}(A))$ .

Let  $\text{ind } \mathcal{D}_2(A)$  be the set of isomorphism classes of all indecomposable objects in  $\mathcal{D}_2(A)$  and  $\text{ind}(A)$  be the set of isomorphism classes of all indecomposable objects in  $\text{mod}(A)$ . We know that  $\text{ind } \mathcal{D}_2(A) = \text{ind}(A) \cup (\text{ind}(A)[1])$  by [H1].

For any  $\alpha \in K_0(\text{mod } A)$ , let  $\mathbb{E}(A, \alpha)$  be the module variety on the dimension vector  $\alpha$ . Let  $M_G(\alpha)$  be the  $\mathbb{C}$ -space of  $G_\alpha$ -invariant constructible functions over  $\mathbb{E}(A, \alpha)$ . We can define a convolution product over  $M_G(A) = \bigoplus_{\alpha \in K_0(\text{mod } A)} M_G(\alpha)$  such that

$$1_{\mathcal{O}_1} * 1_{\mathcal{O}_2}(M) = \chi(\{0 \subset M_1 \subset M \mid M_1 \cong Y, M/M_1 \cong X \text{ for some } X \in \mathcal{O}_1, Y \in \mathcal{O}_2\}).$$

for any  $M \in \mathbb{E}(A, \alpha_1 + \alpha_2)$  and any  $G_\alpha$ -invariant constructible subsets  $\mathcal{O}_1 \subset \mathbb{E}(A, \alpha_1)$  and  $\mathcal{O}_2 \subset \mathbb{E}(A, \alpha_2)$  (see [Rie] and [L5]).

A function is called indecomposable if all points in its support set correspond to indecomposable  $A$ -modules. Let  $I_G(\alpha)$  be the  $\mathbb{C}$ -space of  $G_\alpha$ -invariant indecomposable constructible functions over  $\mathbb{E}(A, \alpha)$ . We set

$$\mathfrak{n}_A = \bigoplus_{\alpha \in K_0(\text{mod } A)} I_G(\alpha)$$

The convolution induces a well-defined Lie bracket over  $\mathfrak{n}_A$  (see [Rie] and [DXX]). For any  $f \in I_G(\alpha)$ , there is an equivalence class  $\hat{f}$  over  $\mathcal{P}_2(A, \alpha)$  by the embedding  $\text{mod } A \hookrightarrow \mathcal{K}_2(\mathcal{P}(A))$ . It is not difficult to prove that the morphism:

$$\begin{aligned} \mathfrak{n}_A &\rightarrow \mathfrak{g} \\ 1_{\mathcal{O}} &\mapsto \hat{1}_{\mathcal{O}} \end{aligned}$$

is an injective Lie algebra homomorphism. So  $\mathfrak{n}_A$  is isomorphic to a Lie subalgebra (denoted by  $\mathfrak{n}^+$ ) of  $\mathfrak{g}$ , which deduces a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  and  $\mathfrak{n} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$  where  $\mathfrak{n}^- := \mathfrak{n}_A[1]$  is generated by  $\hat{1}_{\mathcal{O}[1]}$  with  $1_{\mathcal{O}} \in \mathfrak{n}_A$ . The triangular decomposition  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  and the non-degenerated bilinear form  $(-|-)_{|\mathfrak{n}_A \oplus \mathfrak{n}_A[1]}$  guarantee that  $\mathfrak{g}$  is a generalized Kac-Moody Lie algebra by [B]. The generators and generating relations can be constructed by the following process.

6.2. First, Let  $\mathcal{LE}(A)$  be the Lie subalgebra of  $\mathfrak{g}$  generated by all  $\hat{1}_{\mathcal{O}_X}$  which  $\mathcal{O}_X$  is the orbit of  $X$  with  $X$  exceptional object in  $\mathcal{D}_2(A)$ . It can be proved that  $\mathcal{LE}(A)$  is the corresponding Kac-Moody Lie algebra (see [PX3] Theorem 4.7). Let  $\mathcal{LE}(A)^\pm = \mathcal{LE}(A) \cap \mathfrak{n}^\pm$  be its Lie subalgebras of positive and negative parts, respectively. We set  $\mathfrak{g}_0^+(A) = \mathcal{LE}(A)^+$  and  $\mathfrak{g}_0^-(A) = \mathcal{LE}(A)^-$ . We shall construct the Lie subalgebra  $\mathfrak{g}_m^\pm(A)$  of  $\mathfrak{n}^\pm(A)$  and the Lie subalgebra  $\mathfrak{g}_m(A)$  of  $\mathfrak{g}(A)$  for  $m \geq 0$ .

Let  $m \geq 1$  and suppose  $\mathfrak{g}_{m-1}^\pm(A)$  and  $\mathfrak{g}_{m-1}(A) = \mathfrak{g}_{m-1}^- \oplus \mathfrak{h} \oplus \mathfrak{g}_{m-1}^+$  have been constructed. We let  $\pi_m \in \mathbb{N}[I]$  have the smallest trace such that  $\mathfrak{g}_{m-1}^\pm(A)_{\pi_m} \neq \mathfrak{n}^\pm(A)_{\pi_m}$ . Then we define

$$L_{\pi_m}^+ = \{x^+ \in \mathfrak{n}^+(A)_{\pi_m} \mid (x^+ | \mathfrak{g}_{m-1}^-(A)_{\pi_m}) = 0\}$$

and

$$L_{\pi_m}^- = \{y^- \in \mathfrak{n}^-(A)_{\pi_m} \mid (\mathfrak{g}_{m-1}^+(A)_{\pi_m} | y^-) = 0\}$$

We now denote by  $\mathfrak{g}_m^\pm(A)$  the Lie subalgebra of  $\mathfrak{n}^\pm(A)$  generated by  $\mathfrak{g}_{m-1}^\pm(A)$  and  $L_{\pi_m}^\pm$ , respectively. Set  $\mathfrak{g}(A)_m = \mathfrak{g}_m^+(A) \oplus \mathfrak{h} \oplus \mathfrak{g}_m^-(A)$ . As a conclusion, we obtain the chains of Lie subalgebra of  $\mathfrak{n}^\pm(A)$  and of  $\mathfrak{g}(A)$ :

$$\mathfrak{g}^\pm(A)_0 \subset \mathfrak{g}^\pm(A)_1 \subset \cdots \subset \mathfrak{g}^\pm(A)_m \subset \cdots \subset \mathfrak{n}^\pm,$$

$$\mathfrak{g}(A)_0 \subset \mathfrak{g}(A)_1 \subset \cdots \subset \mathfrak{g}(A)_m \subset \cdots \subset \mathfrak{g}.$$

For  $m \geq 1$ , let  $\eta_m = \dim_{\mathbb{C}} L_{\pi_m}^+ = \dim_{\mathbb{C}} L_{\pi_m}^-$ . All  $\pi_m$  lie in the fundamental set, that is,  $(\pi_m, i) \leq 0$  for  $i \in I$  and  $m \geq 1$  (see [DX]). Depending on non-degeneracy of

$(-|-) \mid_{\mathbf{n}^+ \times \mathbf{n}^-}$ , we can construct a basis  $\{E_p(m) \mid 1 \leq p \leq \eta_m\}$  of  $L_{\pi_m}^+$  and a basis  $\{F_p(m) \mid 1 \leq p \leq \eta_m\}$  of  $L_{\pi_m}^-$  such that

$$(E_p(m), F_q(n)) = \delta_{pq} \delta_{mn}.$$

Now we set  $e_i = E_i(0) = \hat{1}_{\mathcal{O}_{S_i}}$  and  $f_i = F_i(0) = \hat{1}_{\mathcal{O}_{S_i[1]}}$  for  $i \in I$ , where  $S_i$  is the simple  $A$ -module at the vertex  $i$ . Let  $e_j = E_p(m)$  and  $f_j = F_p(m)$  for  $j = (\pi_m, p) \in J$  where  $J = \{(\pi_m, p) \mid 1 \leq p \leq \eta_m\}$ . If  $j = (\pi_m, p) \in J$  we denote  $h_j = h_{\pi_m}$ . Following the above construction,  $\mathfrak{g}$  is the Lie algebra generated by  $e_i, f_i$  and  $h_i$  for  $i \in I \cup J$ . We define  $\omega(e_i) = f_i$  for  $i \in I \cup J$  and  $\omega(h_i) = -h_i$  for  $i \in I$ . The property of non-degeneracy of  $(-|-) \mid_{\mathbf{n}^+ \times \mathbf{n}^-}$  implies that  $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$  is an involution.

The triangular decomposition  $\mathfrak{g} = \mathbf{n}^- \oplus \mathfrak{h} \oplus \mathbf{n}^+$ , the non-degeneracy and invariance of bilinear form  $(-|-)$  guarantee the generators satisfy the generating relations of the generalized Kac-Moody Lie algebra (see [SV] and [DX]). Hence, we have the following theorem by [B]:

**Theorem 6.1.** *Let  $A = \mathbb{C}Q$ . The Lie algebra  $\mathfrak{g}$  is a generalized Kac-Moody Lie algebra with the following generating relations:*

- (1)  $[h_i, h_j] = 0$  for any  $i, j \in I$ ;
- (2)  $[h_i, e_j] = (i, j)e_j$  for any  $i \in I$  and  $j \in I \cup J$ ;
- (3)  $[h_i, f_j] = -(i, j)f_j$  for any  $i \in I$  and  $j \in I \cup J$ ;
- (4)  $[e_i, f_j] = \delta_{ij}h_j$  for any  $i, j \in I \cup J$ ;
- (5)  $(\text{ade}_i)^{1-a_{ij}}e_j = (\text{adf}_i)^{1-a_{ij}}f_j = 0$  for any  $i \in I, j \in I \cup J$  and  $i \neq j$ ;
- (6)  $[e_i, e_j] = [f_i, f_j] = 0$  if  $a_{ij} := (i, j) = 0$  for  $i, j \in I \cup J$ .

Moreover, the Lie subalgebra  $\mathfrak{g}_0(A)$  generated by  $e_i, f_i$  and  $h_i$  for  $i \in I$  is the derived Kac-Moody Lie algebra with Cartan datum  $(I, (-, -))$ , where  $(-, -)$  is the symmetric Euler bilinear form of  $Q$ .

6.3. In the following part, we will suppose  $Q$  is a Dynkin or tame quiver. An explicit construction of the positive part of affine Kac-Moody algebra has been given in [FMV] via Hall algebra approach. We will verify that, if we apply our result to the case the derived category of representations of a tame quiver, the main result in [FMV] can be extended easily to give a global realization of the affine Kac-Moody algebra. We remark here that the similar result has been obtained by [LP] in a different way.

6.3.1 As showed in [GM] and [XZZ], BGP-functor( see [BGP]) can be defined over root category  $\mathcal{D}_2(A)$ , denoted by:

$$\mathcal{D}_2(A) \xrightarrow{H_m(\mathcal{S}_a^+)(H_m(\mathcal{S}_a^-))} \mathcal{D}_2(\sigma_a A)$$

where  $a$  is a source (sink) of quiver  $Q$  and  $\sigma_a A = \mathbb{C}(\sigma_a Q)$ . In particular,

$$H_m(\mathcal{S}_a^+)(S_a) = S'_a[1]$$

where  $S_a$  and  $S'_a$  are the simple modules at the vertex  $a$  in  $\text{mod } A$  and  $\text{mod } \sigma_a A$ , respectively. The explicit construction as follows (see Ex 4.6 in [GM]). Let  $M^\bullet = (M_n, \partial_n) \in \mathcal{D}_2(A)$  where  $M_n = (\mathbb{C}^{\alpha_n}, x_n)$  with dimension vector  $\alpha_i$  for  $n \in \mathbb{Z}$ . Recall  $\mathbb{C}^{\alpha_n} = \bigoplus_{i \in Q_0} \mathbb{C}^{d_i(\alpha_n)}$  and  $x_n = (x_{n,\alpha})_{\alpha \in Q_1}$  with  $x_{n,\alpha} \in \text{Hom}(\mathbb{C}^{d_{s(\alpha)}(\alpha_n)}, \mathbb{C}^{d_{t(\alpha)}(\alpha_n)})$ . Of course,  $M_n = M_{n+2}$ . Define  $H_m(S_i^+)(M^\bullet) = (N_n, d_n) \in \mathcal{D}_2(\sigma_i A)$  as follows. Assume that  $N_n = (\mathbb{C}^{\beta_n}, y_n)$  with dimension vector  $\beta_n$ . Then

$$\mathbb{C}^{d_i(\beta_n)} = \mathbb{C}^{d_i(\alpha_n)} \text{ for any } i \neq a,$$

and

$$\mathbb{C}^{d_a(\beta_n)} = \left( \bigoplus_{(ab) \in Q_1} \mathbb{C}^{d_b(\alpha_n)} \right) \oplus \mathbb{C}^{d_a(\alpha_{n+1})}$$

and for  $\beta \in Q_1$ ,

$$y_{n,\beta} = \begin{cases} x_{n,\beta}, & \text{if } t(\beta) \neq a, \\ \text{the embedding}, & \text{otherwise.} \end{cases}$$

Recall that  $\mathcal{LE}(Q)$  is Lie subalgebra of  $\mathfrak{g}$  generated by the character functions of exceptional objects in the root category. This functor induces the homeomorphism between  $\mathcal{P}_2(A)$  and  $\mathcal{P}_2(\sigma_i A)$  so that we have the following isomorphism between the corresponding Lie algebras:

$$\tilde{\varphi}_i : \mathcal{LE}(A) \rightarrow \mathcal{LE}(\sigma_i A)$$

By Theorem 6.1 we have the following canonical isomorphism:

$$\phi : \mathfrak{g}(C) \rightarrow \mathcal{LE}(A)$$

fixing the generators  $e_i$  and  $f_i$  for  $i \in I$ , where  $C = (I, (-, -))$  is the Cartan datum of  $Q$ ,  $\mathfrak{g}(C)$  is the derived Kac-Moody Lie algebra corresponding to  $C$ . Moreover, we have the following commutative diagram (see [XZZ]):

$$\begin{array}{ccc} \mathfrak{g}(C) & \xrightarrow{\tilde{s}_i} & \mathfrak{g}(C) \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{LE}(A) & \xrightarrow{\tilde{\varphi}_i} & \mathcal{LE}(\sigma_i A) \end{array}$$

where  $\tilde{s}_i := \exp(\text{ade}_i)\exp(\text{ad}(-f_i))\exp(\text{ade}_i)$  is the isomorphism map defined in [K3].

**6.3.2** In order to give the concrete expression of Chevalley basis in  $\mathfrak{g}(C)$  by the above isomorphism  $\phi$  in  $\mathcal{LE}(A)$ , we will consider Lie algebra based on the Euler cocycle. Let  $Q$  is a tame quiver and  $R$  be the root system of the quiver  $Q$ . Let  $\delta$  be the minimal imaginary root of  $R_+$ . We denote by  $\mathfrak{g}^\epsilon(C)$  the following  $R$ -graded  $\mathbb{C}$ -linear space:

$$\mathfrak{g}^\epsilon(C) = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{n}_\alpha^\epsilon(C),$$

such that

- for each real root  $\alpha$  a one-dimensional  $\mathbb{C}$ -linear space  $\mathfrak{n}_\alpha^\epsilon(C) = \mathbb{C}\tilde{e}_\alpha$  with generator  $\tilde{e}_\alpha$ ,
- for each imaginary root  $n\delta$  a  $\mathbb{C}$ -linear space  $\mathfrak{n}_{n\delta}^\epsilon(C) = \mathbb{C}[I]/\mathbb{C}\delta$ , where we consider  $\delta$  as an element of  $R_+ \subset \mathbb{Z}[I] \subset \mathbb{C}[I]$ . For  $h \in \mathbb{C}[I]$  we denote by  $h(n)$  the image of  $h$  under the natural projection map  $\mathbb{C}[I] \rightarrow \mathfrak{n}_{n\delta}^\epsilon(C)$ .

The space  $\mathfrak{g}^\epsilon(Q)$  is equipped with the bilinear bracket

$$\begin{aligned}
(6.1) \quad [\tilde{e}_\alpha, \tilde{e}_\beta] &= \begin{cases} \epsilon(\alpha, \beta)\tilde{e}_{\alpha+\beta} & \text{if } \alpha + \beta \in R^{\text{re}}, \\ \epsilon(\alpha, \beta)\alpha(k) & \text{if } \alpha + \beta = k\delta, \\ h_\alpha & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \\
[h(n), \tilde{e}_\alpha] &= -[\tilde{e}_\alpha, h(n)] = \epsilon(n\delta, \alpha)(h, \alpha)\tilde{e}_{\alpha+n\delta}, \\
[h(n), h'(-n)] &= n(h, h')c, \quad c \text{ is the center element} \\
[h(n), h(m)] &= 0, \quad \text{if } n + m \neq 0
\end{aligned}$$

where  $h \in \mathbb{C}[I]$ ,  $\epsilon$  is the Euler cocycle. The following is well-known.

**Theorem 6.2.** *The  $\mathbb{C}$ -linear space  $\mathfrak{g}^\epsilon(Q)$  equipped with the above bracket is isomorphic to the derived affine Kac-Moody algebra of type  $C$ .*

Next, we describe the structure of  $\mathcal{D}_2(A)$  (See [H1]) for  $A = \mathbb{C}Q$  of tame quiver  $Q$ . We know  $\text{ind } \mathcal{D}_2(A) = \text{ind } A \cup \text{ind } A[1]$ . The AR-quiver of  $\mathcal{D}_2(A)$  consists of two components of form  $\mathbb{Z}Q$  and two families of orthogonal tubes indexed by  $\mathbb{P}^1(\mathbb{C})$ . We denote the two families of tubes by  $\mathbb{T}_Q = \{T_z \mid z \in \mathbb{P}^1(\mathbb{C})\}$  and by  $N_z$  the period of  $T_z$  and  $\mathbb{T}_Q[1] = \{T_z[1] \mid z \in \mathbb{P}^1(\mathbb{C})\}$ . Let  $L \subset \mathbb{P}^1(\mathbb{C})$  be such that the tube  $T_z$  is of period  $N_z > 1$  for any  $z \in L$ . Then  $|L| \leq 3$ . By [DR], we may fix an embedding:  $\pi : \mathcal{D}_2(K) \rightarrow \mathcal{D}_2(A)$  such that  $\pi(\mathbb{T}_K) \subset \mathbb{T}_Q$  and  $\pi(\mathbb{T}_K[1]) \subset \mathbb{T}_Q[1]$  for the Kronecker quiver  $K$ . It induces the Hall map  $\pi_*$  from  $M_{GT}(K)$  to  $M_{GT}(A)$ .

Let  $\hat{1}_{(n,n)} \in I_{GT}(K, (n, n))$  be the equivalence class over  $\mathcal{P}_2(K, (n, n))$  of the characteristic function on the constructible subset of all indecomposable objects in  $\text{mod } K$  with dimension vector  $(n, n)$  for  $n \in \mathbb{Z}$ . We denote by  $E_0(n)$  the image of  $\hat{1}_{(n,n)}$  under  $\pi_*$ .

Let  $M_{i,1,z}$  be the regular simple objects in  $T_z$  and  $M_{i,l,z}$  be the indecomposable objects in  $T_z$  with regular length  $l$  for  $i \in \mathbb{Z}/N_z\mathbb{Z}$  such that the regular socle of  $M_{i,l,z}$  is  $M_{i,1,z}$ . Set  $M_{i,-l,z} = M_{i,l,z}[1]$  for  $l > 0$ . Let  $E_{i,l,z} := 1_{\mathcal{O}_{M_{i,l,z}}}$  be the characteristic function of the orbit of  $M_{i,l,z}$ . If  $|l| \not\equiv 0 \pmod{N_z}$ , then  $\underline{\dim} M_{i,l,z} \in R^{\text{re}}$  and  $E_{i,l,z} \in \mathcal{LE}(A)$ . If  $l = nN_z$ , then  $\underline{\dim} M_{i,l,z} = n\delta$  and  $E_{i+1,l,z} - E_{i,l,z} \in \mathcal{LE}(A)$  (See [FMV] and [PX1]).

We define the function  $\xi : R \rightarrow \pm 1$  by

$$\xi(\alpha) = (-1)^{(1+\dim_{\mathbb{C}} \text{End } M)}$$

for indecomposable object  $M$  with  $\underline{\dim} M = \alpha$ .

For any function  $f_\alpha \in I_{GT}(\mathcal{D}_2(A))$ , set  $\tilde{f}_\alpha = \xi(\alpha)f_\alpha$ .

Now we are ready to give the generalization of the main theorem in [FMV]. By using the reflection functor defined for  $\mathcal{D}_2(A)$  and following the method in [FMV] we eventually obtain the following result.

**Theorem 6.3.** *Let  $Q$  be any tame quiver without oriented cycles. The following map completely describes an isomorphism  $\mathfrak{g}^\epsilon(C) \rightarrow \mathcal{LE}(A)$ :*

$$\begin{aligned}
\Xi^\epsilon(\tilde{e}_\alpha) &= \tilde{E}_\alpha \text{ for any } \alpha \in R^{\text{re}}, \\
\Xi^\epsilon(\alpha_{i,z}(n)) &= \tilde{E}_{i,nN_z,z} - \tilde{E}_{i+1,nN_z,z}, \\
\Xi^\epsilon(\alpha_0(n)) &= -\tilde{E}_0(n) \\
\Xi^\epsilon(\beta) &= h_\beta, \Xi^\epsilon(c) = h_\delta.
\end{aligned}$$

for any  $n \in \mathbb{Z}$  and  $\beta \in \mathbb{Z}[I]$ .

Obviously, the set:

$$\{\tilde{E}_\alpha\}_{\alpha \in R^{\text{re}}} \cup \{\tilde{E}_{i,nN_z,z} - \tilde{E}_{i+1,nN_z,z}\}_{i \in \mathbb{Z}/N_z\mathbb{Z}, z \in L, n \in \mathbb{Z}} \cup \{\tilde{E}_0(n)\}_{n \in \mathbb{Z}} \cup \{h_i\}_{i \in I}$$

provides a  $\mathbb{Z}$ -basis of  $\mathcal{LE}(A)$ .

## REFERENCES

- [ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press (1995).
- [B] R. Borcherds, *Generalized Kac-Moody algebras*, J.Algebra **115** (1988), 501-512.
- [BD] V. Bekkert and Y. Drozd, *Tame-wild dichotomy for derived categories*, arXiv:math.RT/0310352.
- [BGP] J. Bernstein, I. Gelfand, and V. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspehi Mat. Nauk **28** (1973), no. 2(170), 19-33.
- [CBV] W. Crawley-Boevey and M. Van den Bergh, *Absolutely indecomposable representations and Kac-Moody Lie algebras (with an appendix by Hiraku Nakajima)*, Invent. Math. **155** (2004), 537-559.
- [CS] C. de Concini and E. Strickland, *On the variety of complexes*, Advances in Math. **41** (1981), 57-77.
- [D] H. Derksen, *Quotients of algebraic group actions*, In: A. van den Essen (ed.), *Automorphisms of Affine Spaces*, Kluwer Academic Publishers, the Netherlands, (1995) 191-200.
- [Di] A. Dimca, *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, 2004.
- [Dr] V. G. Drinfeld, *Quantum groups*, Proc. Int. Congr. Math. Berkeley 1986, Vol **1**, Amer. Math. Soc., (1988) 798-820.
- [DR] V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **6** (1976), no. 173.
- [DX] B. Deng and J. Xiao, *On double Ringel-Hall algebras*, J.Algebra, **251**, (2002) 110-149.
- [DXX] M. Ding, J. Xiao and F. Xu, *Realizing enveloping algebras via varieties of modules*, to appear in Acta Math. Sinica.
- [FMV] I. Frenkel, A. Malkin and M. Vybornov, *Affine Lie algebras and tame quivers*, Selecta Math.(N.S.) **7** (2001) 1-56.
- [G] P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71-103; correction, ibid. **6** (1972), 309.
- [Gr] J. A. Green, *Hall algebras, hereditary algebras and quantum groups*, Invent.Math.**120** (1995), 361-377.
- [GM] S. I. Gelfand, and Y. I. Manin, *Methods of Homological Algebra*, Springer 1996.
- [H1] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, LMS **119**, Cambridge University Press, 1988.
- [H2] D. Happel, *A characterization of hereditary categories with tilting object*, Invent. Math. **144** no. 2 (2001), 381-398.
- [Hu] A. Hubery, *From triangulated categories to Lie algebras: a theorem of Peng and Xiao*, To appear in *Proceedings of the Workshop on Representation Theory of Algebras and related Topics* (Querétaro, 2004).
- [I] B. Iversen, *Cohomology of Sheaves*, Springer, 1986.
- [J] M. Jimbo, *A q-difference analogue of  $U(\mathfrak{g})$  and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63-69.
- [Jo1] D. Joyce, *Constructible functions on Artin stacks*, J. London Math. Soc. **74**(2006), 583-606.
- [Jo2] D. Joyce, *Configurations in abelian categories II: Ringel-Hall algebras*, Advances in Mathematics **210** (2007), 635-706.
- [JSZ] B. T. Jensen, X. Su and A. Zimmermann, *Degeneration for derived categories*, Journal of Pure and Applied Algebra, **198** (2005), 281-295.
- [K1] V. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. **56** (1980), 57-92.
- [K2] V. Kac, *Root systems, representations of quivers and invariant theory*, Lecture Notes in Mathematics, Vol. **996**, Springer, Berlin, 1982, 74-108.

- [K3] V. Kac, *Infinite dimensional Lie algebras*, third edition, Cambridge University Press 1990.
- [Ka] M. Kapranov, *Heisenberg doubles and derived categories*, J. Algebra **202** (1998), 712-744.
- [KS] M.Kashiwara and Y.Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9-36.
- [Ke] B. Keller, *On triangulated orbit categories*, Doc.Math. **10** (2005), 551-581
- [KP] H. Kraft and V. Popov, *Semisimple group actions on the three dimensional affine space are linear*, Comment. Math. Helv. **60** (1985), 466-479.
- [LP] Y. Lin and L. Peng, *2-extended affine Lie algebras and tubular algebras*, Advances in Math. **196** (2005), 487-530.
- [L1] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447-498.
- [L2] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), no. 2, 365-421.
- [L3] G. Lusztig, *Affine quivers and canonical bases*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 76, 111-163.
- [L4] G. Lusztig, *Introduction to quantum groups*, Progr. Math. **110**, Birkhauser, 1993.
- [L5] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Advances in Math. **151** (2000), 129-139.
- [L6] G. Lusztig, *Constructible functions on varieties attached to quivers*, in Studies in memory of Issai Schur, 177-223, Progress in Mathematics **210**, Birkhäuser 2003.
- [Mac] R.D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. Math. **100**, 423-432 (1974).
- [Mum] D. Mumford, *Algebraic Geometry I: complex Projective Varieties*, Springer-Verlag, Berlin Heidelberg New York, 1976.
- [N] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), 515-560.
- [Pa] Y. Palu, *Cluster character II: a multiplication formula*, arXiv:0903.3281.
- [PX1] L. Peng and J. Xiao, *A realization of affine Lie algebras of type  $\tilde{A}_{n-1}$  via the derived categories of cyclic quivers*, CMS Conf. Proc. **18**, 539-554, Amer. Math. Soc., Providence, RI, 1996.
- [PX2] L. Peng and J. Xiao, *Root categories and simple Lie algebras*, J. Algebra **198** (1997), no. 1, 19-56.
- [PX3] L. Peng and J. Xiao, *Triangulated categories and Kac-Moody Lie algebras*, Invent. Math. **140** (2000), 563-603.
- [Rick1] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2), **39(3)**, 436-456, 1989.
- [Rick2] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2), **43(1)**, 37-48, 1991.
- [Rie] Ch. Riedtmann, *Lie algebras generated by indecomposables*, J. Algebra **170** (1994), no. 2, 526-546.
- [R1] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583-592.
- [R2] C. M. Ringel, *Hall polynomials for the representation-finite hereditary algebras*, Adv. Math. **84** (1990), no. 2, 137-178.
- [R3] C. M. Ringel, *Lie algebras arising in representation theory*, in: "Representations of Algebras and Related Topics", London Math.Soc.LNS **168**, Cambridge Univ. Press (1992), 284-291.
- [Ro] M. Rosenlicht, *A Remark on quotient spaces*, An. Acad. Brasil. Ciênc. **35** (1963), 487-489.
- [Sch] A. Schofield, *Notes on constructing Lie algebras from finite-dimensional algebras*, Preprint, 1991.
- [SV] B. Sevenhant and M. Van den Bergh, *A relation between a conjecture of Kac and the structure of the Hall algebra*, J. Pure and Appl. Algebra. **160** (2001), 319-332.
- [SHZ] M. Saorin and B. Huisgen-Zimmermann, *Geometry of chain complexes and outer automorphisms under derived equivalence*, Trans. AMS. **353** (2001), 4757-4777.
- [T] B. Toën, *Derived Hall algebras*, Duke Math. J. **135** (2006), no. 3, 587-615.

- [Xu] F. Xu, *On the cluster multiplication theorem for acyclic cluster algebras*, Trans. Amer. Math. Soc. **362** (2010), 753-776.
- [XX] J.Xiao and F. Xu, *Hall algebra associated to a triangulated category*, Duke Math. J. **143** (2008), no. 2, 357-373.
- [XZZ] J.Xiao, G.Zhang and B.Zhu, *BGP Reflection functor over Root Category*, Science in China (Series A) **35(4)** (2005), 375-386.

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